Inextensible flows of curves in three-dimensional light cone

H. S. Abdel-Aziz[1], **M. Khalifa Saad**[2] **and Haytham. A. Ali**[3]

[1]Department of Mathematics
Sohag University
82524 Sohag, Egypt
habdelaziz2005@yahoo.com

[2]Department of Mathematics
Sohag University
82524 Sohag, Egypt
mohamed_khalifa77@science.sohag.edu.eg

[1]Department of Mathematics
Sohag University
82524 Sohag, Egypt
haytham.ali88@yahoo.com

ABSTRACT

In this paper, we study inextensible flows of spacelike and lightlike (null) curves in 3-dimensional light cone Q^3 of Minkowiski 4-space E_1^4. We introduce the necessary and sufficient conditions for inextensible flows of these curves which can be expressed as partial differential equations involving the curvatures.

Keywords: Spacelike and null curves, Frenet frame, Inextensible flow, Minkowski space.

2000 Mathematics Subject Classification: 53A35, 53A17, 51B20.

1 Introduction

Many nonlinear phenomena in different fields of science such as physics, chemistry and biology are described by dynamics of shapes, such as curves and surfaces. The evolution of curve and surface has significant applications in computer vision and image processing (Sapiro, 2001). According to the great importance of the motion of curves and its evolution equations in physical and engineering applications (Yeneroğlu et al., 2010; Gürbüz, 2009), so we interest here with this subject. The flow of a curve is said to be inextensible if its arclength is preserved. There have been number of studies in the literature on plane curve flows, particularly on evolving curves in the direction of their curvature vector field. In (Gage, 1986) the auther study applications of inextensible curve flows and study shrinking of closed plane curves to a circle via the heat equation but in these studies, the flow is not inextensible.

In this work, we study inextensible flows of spacelike and null curves in $Q^3 \subset E_1^4$. We derive the evolution equations for inextensible flows for these curves. Necessary and sufficient conditions for an inextensible curve flow were obtained and expressed as partial differential equations involving the curvatures of the considered curves.

2 Fundamental meanings on Minkowski 4-space

Let E_1^4 be the 4-dimensional pseudo-Euclidean space with the metric

$$g(\mathbf{x},\mathbf{y}) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4,$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4), \mathbf{y} = (y_1, y_2, y_3, y_4) \in E_1^4$. Let M be a submanifold of E_1^4, if the pseudo-Riemannian metric g of E_1^4 induces a pseudo-Riemannian metric g (respectively, a Riemannian metric, a degenerate quadratic form) on M, then M is called a timelike (respectively, spacelike, degenerate) submanifold of E_1^4. Let c be a fixed point in E_1^4 and $r > 0$ be a constant, the pseudo-Riemannian sphere is defined by

$$S_1^3(c,r) = \{\mathbf{x} \in E_1^4 : g(\mathbf{x}-c, \mathbf{x}-c) = r^2\};$$

the pseudo-Riemannian hyperbolic space is defined by

$$H_1^3(c,r) = \{\mathbf{x} \in E_2^4 : g(\mathbf{x}-c, \mathbf{x}-c) = -r^2\};$$

the pseudo-Riemannian light cone (quadric cone) is defined by

$$Q_1^3(c,r) = \{\mathbf{x} \in E_1^4 : g(\mathbf{x}-c, \mathbf{x}-c) = 0\}.$$

The spaces $S_1^3(c,r), H_1^3(c,r)$ and $Q_1^3(c)$ are called pseudo-Riemannian space forms and the point c represents the center for each of them. When $c = 0$, we simply denote $Q_1^3(0)$ by Q^3 and call it the lightlike cone (or simply the light cone).

Let E_1^4 be the 4-dimensional Minkowski space and Q^3 be the lightlike cone in E_1^4. A vector $v \neq 0$ in E_1^4 is called spacelike, timelike or lightlike, if $\langle v,v\rangle > 0$, $\langle v,v\rangle < 0$ or $\langle v,v\rangle = 0$, respectively. Similarly, an arbitrary curve $\alpha = \alpha(s)$ can be locally spacelike, timelike or null, if all of its velocity $\alpha'(s)$ are respectively spacelike, timelike or null(O'neill, 1983).

Spacelike curves in Q^3

Let $\delta = \delta(s) : I \to Q^3 \subset E_1^4$ be a spacelike curve, to each unit speed δ one can associate an asymptotic orthonormal frame $\{\delta, T, N, B\}$, then the Frenet formulas of δ read (Liu, 2004; Liu and Meng, 2011)

$$\left.\begin{array}{c} \delta'(s) = T(s), \\ T'(s) = \kappa(s)\delta(s) + \sigma(s)N(s) - B(s), \\ N'(s) = \tau(s)\delta(s) - \sigma(s)T(s), \\ B'(s) = -\kappa(s)T(s) - \tau(s)N(s), \end{array}\right\} \tag{2.1}$$

with the following conditions:

$$\begin{aligned} \langle\delta,\delta\rangle &= \langle B,B\rangle = \langle\delta,T\rangle = \langle\delta,N\rangle = \langle B,T\rangle = \langle B,N\rangle = \langle T,N\rangle = 0, \\ \langle\delta,B\rangle &= \langle T,T\rangle = \langle N,N\rangle = 1, \end{aligned}$$

where the functions $\kappa(s)$, $\tau(s)$ and $\sigma(s)$ are called the first, second and third curvatures of $\delta(s)$ and which are defined as

$$\kappa(s) = -\frac{1}{2}\langle\delta''(s), \delta''(s)\rangle,\ \tau^2(s) = \langle\delta''(s), \delta''(s)\rangle - 4\kappa^2(s).$$

Lightlike curves in Q^3

Let $\gamma(s)$ be a null curve on Q^3, then there exists a *Natural Frenet frame* $\{\gamma, e_1, e_2, e_3\}$ satisfying the following equations:

$$\left.\begin{array}{c} \gamma'(s) = e_1(s), \\ e_1'(s) = h(s)e_1(s) + \kappa_1(s)\gamma(s), \\ e_2'(s) = h(s)e_2(s) + \kappa_2(s)\gamma(s) - e_3(s), \\ e_3'(s) = -\kappa_2(s)e_1(s) - \kappa_1(s)e_2(s), \end{array}\right\} \tag{2.2}$$

where $e_1,\ e_2$ and e_3 are the tangent vector, the unique null transversal vector to e_1 and the unique null transversal vector to $\gamma(s)$.

Here the following conditions are satisfied:

$$\begin{aligned} \langle\gamma, \gamma\rangle &= \langle\gamma, e_1\rangle = \langle\gamma, e_2\rangle = \langle e_1, e_1\rangle = \langle e_1, e_3\rangle = \langle e_2, e_2\rangle \\ &= \langle e_3, e_3\rangle = \langle e_2, e_3\rangle = 0,\ \langle\gamma, e_3\rangle = \langle e_1, e_2\rangle = 1, \end{aligned}$$

where $h(s)$, $\kappa_1(s)$ and $\kappa_2(s)$ are the curvatures of $\gamma(s)$ which are defined by (Sun and Pei, 2015)

$$h(s) = \langle e_1'(s), e_2(s)\rangle, \kappa_1(s) = \langle e_1'(s), e_3(s)\rangle, \kappa_2(s) = \langle e_2'(s), e_3(s)\rangle.$$

2.1 Flows of inextensible curves

In what follows, we present some basic notions on inextensible flows of curves as follows. Assume that $\Gamma : [0, l] \times [0, t^{\infty}] \to E_1^4$ is the family of differentiable curves in the light cone Q^3, where l is the arclength of the initial curve. Let u be the curve parametrization variable, $0 \le u \le l$. If the speed curve in $Q^3 \subset E_1^4$ is given by $v = \left\|\frac{\partial\Gamma}{\partial u}\right\|$, then the arclength of Γ is given as a function of u by

$$s(u) = \int_0^u \left\|\frac{\partial\Gamma}{\partial u}\right\| du = \int_0^u v\ du,$$

where

$$\left\|\frac{\partial\Gamma}{\partial u}\right\| = \sqrt{\left|\langle\frac{\partial\Gamma}{\partial u}, \frac{\partial\Gamma}{\partial u}\rangle\right|}. \tag{2.3}$$

The operator $\frac{\partial}{\partial u}$ is given by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u},$$

and the arclength parameter is $ds = vdu$ (for a review of curve theory) (Latifi and Razavi, 2008).

Definition 2.1. Let Γ be a family of differentiable spacelike curves in $Q^3 \subset E_1^4$ and $\{\delta, T, N, B\}$ be an asymptotic orthonormal frame of Γ in Minkowski space-time. Any flow of Γ can be expressed as

$$\frac{\partial\Gamma}{\partial t} = \lambda_1\delta + \lambda_2 T + \lambda_3 N + \lambda_4 B,$$

where, λ_i are the i^{th} scalar speed of Γ.

Let the arclength variation be

$$s(u,t)=\int_0^u v du.$$

In E_1^4, the requirement that the spacelike curves not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}\, du=0,$$

for all $u\in[0,l]$. With this in mind we consider the following definition.

Definition 2.2. Let $\Gamma(u,t)$ be a family of differentiable spacelike curves in $Q^3\subset E_1^4$. The curve evolution $\Gamma(u,t)$ and its flow $\frac{\partial\Gamma}{\partial t}$ are said to be inextensible if $\frac{\partial}{\partial t}\left|\frac{\partial\Gamma}{\partial u}\right|=0$.

3 Inextensible flow of a spacelike curve in Q^3

Theorem 3.1. *Let $\frac{\partial\Gamma}{\partial t}=\lambda_1\delta+\lambda_2T+\lambda_3N+\lambda_4B$ be a smooth flow of the family of differentiable spacelike curve Γ in the light cone $Q^3\subset E_1^4$, then we have the following equality*

$$\frac{\partial v}{\partial t}=\left(\frac{\partial\lambda_2}{\partial u}+v\lambda_1-v\lambda_3\sigma-v\lambda_4\kappa\right).$$

Proof. According to (2.3), we have

$$v^2=\langle\frac{\partial\Gamma}{\partial u},\frac{\partial\Gamma}{\partial u}\rangle,$$

$\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since u and t are independent coordinates. We get

$$\begin{aligned}2v\frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\langle\frac{\partial\Gamma}{\partial u},\frac{\partial\Gamma}{\partial u}\rangle=2\langle\frac{\partial\Gamma}{\partial u},\frac{\partial}{\partial u}(\frac{\partial\Gamma}{\partial t})\rangle=2v\langle T,\frac{\partial}{\partial u}(\lambda_1\delta+\lambda_2T+\lambda_3N+\lambda_4B)\rangle\\ &= 2v\left\langle T,\frac{\partial\lambda_1}{\partial u}\delta+v\lambda_1\delta'+\frac{\partial\lambda_2}{\partial u}T+v\lambda_2T'+\frac{\partial\lambda_3}{\partial u}N+v\lambda_3N'+\frac{\partial\lambda_4}{\partial u}B+v\lambda_4B'\right\rangle\end{aligned}$$

By using (2.1), we have

$$\begin{aligned}\frac{\partial v}{\partial t} &= \left\langle T,\left(\frac{\partial\lambda_1}{\partial u}+v\lambda_2\kappa+v\lambda_3\tau\right)\delta+\left(\frac{\partial\lambda_2}{\partial u}+v\lambda_1-v\lambda_3\sigma-v\lambda_4\kappa\right)T\right.\\ &\quad\left.+\left(\frac{\partial\lambda_3}{\partial u}+v\lambda_2\sigma-v\lambda_4\tau\right)N+\left(\frac{\partial\lambda_4}{\partial u}-v\lambda_2\right)B\right\rangle,\end{aligned}$$

which leads to

$$\frac{\partial v}{\partial t}=\left(\frac{\partial\lambda_2}{\partial u}+v\lambda_1-v\lambda_3\sigma-v\lambda_4\kappa\right).$$

From this, the proof is completed. □

Lemma 3.2. *Let $\frac{\partial\Gamma}{\partial t}=\lambda_1\delta+\lambda_2T+\lambda_3N+\lambda_4B$ be a smooth flow of the family of differentiable spacelike curve Γ in the light cone Q^3. The flow is inextensible if and only if*

$$\frac{\partial\lambda_2}{\partial s}=\lambda_3\sigma+\lambda_4\kappa-\lambda_1.$$

Proof. Let us assume that the spacelike curve flow is inextensible then, we have

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u \frac{\partial v}{\partial t}\,du=\int_0^u\left(\frac{\partial \lambda_2}{\partial u}+v\lambda_1-v\lambda_3\sigma-v\lambda_4\kappa\right)\,du=0,$$

which leads to

$$\frac{\partial \lambda_2}{\partial u}+v\lambda_1-v\lambda_3\sigma-v\lambda_4\kappa=0.$$

As we know from the above $\frac{\partial}{\partial s}=\frac{1}{v}\frac{\partial}{\partial u}$, we get

$$\frac{\partial \lambda_2}{\partial s}=\lambda_3\sigma+\lambda_4\kappa-\lambda_1. \tag{3.1}$$

So the proof is completed. □

Lemma 3.3. *Let $\{\delta,T,N,B\}$ be an asymptotic orthonormal frame of a spacelike curves in Q^3 and $\frac{\partial\Gamma}{\partial t}=\lambda_1\delta+\lambda_2T+\lambda_3N+\lambda_4B$ be its flow. The differentiations of $\{\delta,T,N,B\}$ with respect to t are*

$$\begin{aligned}\frac{\partial\delta}{\partial t}&=\psi_2\delta-\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)T+\psi_1N,\\ \frac{\partial T}{\partial t}&=\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2\kappa+\lambda_3\tau\right)\delta+\left(\frac{\partial\lambda_3}{\partial s}+\lambda_2\sigma-\lambda_4\tau\right)N+\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)B,\\ \frac{\partial N}{\partial t}&=\psi_3\delta-\left(\frac{\partial\lambda_3}{\partial s}+\lambda_2\sigma-\lambda_4\tau\right)T-\psi_1B,\\ \frac{\partial B}{\partial t}&=-\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2\kappa+\lambda_3\tau\right)T-\psi_3N-\psi_2B.\end{aligned} \tag{3.2}$$

where

$$\psi_1=\langle\frac{\partial\delta}{\partial t},N\rangle,\quad \psi_2=\langle\frac{\partial\delta}{\partial t},B\rangle \text{ and } \psi_3=\langle\frac{\partial N}{\partial t},B\rangle.$$

Proof. By using definition 2.1, one can write

$$\begin{aligned}\frac{\partial T}{\partial t}&=\frac{\partial}{\partial t}\frac{\partial\Gamma}{\partial s}=\frac{\partial}{\partial s}(\lambda_1\delta+\lambda_2T+\lambda_3N+\lambda_4B)\\ &=\frac{\partial\lambda_1}{\partial s}\delta+\lambda_1\delta'+\frac{\partial\lambda_2}{\partial s}T+\lambda_2T'+\frac{\partial\lambda_3}{\partial s}N+\lambda_3N'+\frac{\partial\lambda_4}{\partial s}B+\lambda_4B'.\end{aligned}$$

From (2.1) and theorem 3.1, we get

$$\begin{aligned}\frac{\partial T}{\partial t}&=\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2\kappa+\lambda_3\tau\right)\delta+\left(\frac{\partial\lambda_2}{\partial s}+\lambda_1-\lambda_3\sigma-\lambda_4\kappa\right)T\\ &\quad+\left(\frac{\partial\lambda_3}{\partial s}+\lambda_2\sigma-\lambda_4\tau\right)N+\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)B.\end{aligned}$$

Using (3.1) to obtain

$$\frac{\partial T}{\partial t}=\left(\frac{\partial\lambda_1}{\partial s}+\lambda_2\kappa+\lambda_3\tau\right)\delta+\left(\frac{\partial\lambda_3}{\partial s}+\lambda_2\sigma-\lambda_4\tau\right)N+\left(\frac{\partial\lambda_4}{\partial s}-\lambda_2\right)B.$$

Since

$$\begin{aligned}
\langle \delta, T\rangle &= 0 \Rightarrow \langle \frac{\partial \delta}{\partial t}, T\rangle = -\left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right), \\
\langle \delta, N\rangle &= 0 \Rightarrow \langle \delta, \frac{\partial N}{\partial t}\rangle = -\psi_1, \\
\langle \delta, B\rangle &= 1 \Rightarrow \langle \delta, \frac{\partial B}{\partial t}\rangle = -\psi_2, \\
\langle T, N\rangle &= 0 \Rightarrow \langle T, \frac{\partial N}{\partial t}\rangle = -\left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right), \\
\langle B, T\rangle &= 0 \Rightarrow \langle \frac{\partial B}{\partial t}, T\rangle = -\left(\frac{\partial \lambda_1}{\partial s} + \lambda_2\kappa + \lambda_3\tau\right), \\
\langle B, N\rangle &= 0 \Rightarrow \langle \frac{\partial B}{\partial t}, N\rangle = -\psi_3,
\end{aligned}$$

and

$$\langle \delta, \frac{\partial \delta}{\partial t}\rangle = \langle T, \frac{\partial T}{\partial t}\rangle = \langle N, \frac{\partial N}{\partial t}\rangle = \langle B, \frac{\partial B}{\partial t}\rangle = 0,$$

we can get

$$\begin{aligned}
\frac{\partial \delta}{\partial t} &= \psi_2\delta - \left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right)T + \psi_1 N, \\
\frac{\partial N}{\partial t} &= \psi_3\delta - \left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right)T - \psi_1 B, \\
\frac{\partial B}{\partial t} &= -\left(\frac{\partial \lambda_1}{\partial s} + \lambda_2\kappa + \lambda_3\tau\right)T - \psi_3 N - \psi_2 B,
\end{aligned}$$

which completes the proof. □

Theorem 3.4. *Suppose that* $\frac{\partial \Gamma}{\partial t} = \lambda_1\delta + \lambda_2 T + \lambda_3 N + \lambda_4 B$ *is inextensible, then the following system of partial differential equations holds:*

$$\frac{\partial \psi_1}{\partial s} = \left(\frac{\partial \lambda_3}{\partial s} - \lambda_4\tau\right) + \sigma\frac{\partial \lambda_4}{\partial s}, \; \frac{\partial \psi_2}{\partial s} = \left(\frac{\partial \lambda_1}{\partial s} + \lambda_3\tau\right) + \kappa\frac{\partial \lambda_4}{\partial s} - \tau\psi_1.$$

Proof. Using Lemma 3.2 to get

$$\frac{\partial}{\partial s}\frac{\partial \delta}{\partial t} = \frac{\partial \psi_2}{\partial s}\delta - \left(\frac{\partial^2 \lambda_4}{\partial s^2} - \frac{\partial \lambda_2}{\partial s}\right)T + \frac{\partial \psi_1}{\partial s}N + \psi_2\delta' - \left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right)T' + \psi_1 N',$$

which gives

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \delta}{\partial t} &= \left(\frac{\partial \psi_2}{\partial s} - \kappa\left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right) + \tau\psi_1\right)\delta + \left(-\left(\frac{\partial^2 \lambda_4}{\partial s^2} - \frac{\partial \lambda_2}{\partial s}\right) + \psi_2 - \sigma\psi_1\right)T \\
&\quad + \left(\frac{\partial \psi_1}{\partial s} - \sigma\left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right)\right)N + \left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right)B. \qquad (3.3)
\end{aligned}$$

Noting that

$$\frac{\partial}{\partial s}\frac{\partial \delta}{\partial t} = \frac{\partial T}{\partial t},$$

hence from (3.2) and (3.3), we obtain

$$\frac{\partial \psi_2}{\partial s} = \left(\frac{\partial \lambda_1}{\partial s} + \lambda_3\tau\right) + \kappa\frac{\partial \lambda_4}{\partial s} - \tau\psi_1, \; \frac{\partial \psi_1}{\partial s} = \left(\frac{\partial \lambda_3}{\partial s} - \lambda_4\tau\right) + \sigma\frac{\partial \lambda_4}{\partial s}.$$

So, the proof is completed. □

Theorem 3.5. *Let $\Gamma(s)$ be a spacelike curve in $Q^3 \subset E_1^4$. Assume that $\frac{\partial \Gamma}{\partial t} = \lambda_1\delta + \lambda_2 T + \lambda_3 N + \lambda_4 B$ is inextensible, then the following system of partial differential equations holds:*

$$\begin{aligned}
\frac{\partial \kappa}{\partial t} &= \frac{\partial^2 \lambda_1}{\partial s^2} + \frac{\partial(\lambda_2\kappa)}{\partial s} + \frac{\partial(\lambda_3\tau)}{\partial s} + \tau\left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right) - \kappa\psi_2 - \sigma\psi_3,\\
\frac{\partial \tau}{\partial t} &= \sigma\left(\frac{\partial \lambda_1}{\partial s} + \lambda_3\tau\right) - \kappa\left(\frac{\partial \lambda_3}{\partial s} - \lambda_4\tau\right) - \psi_2\tau + \frac{\partial \psi_3}{\partial s},\\
\frac{\partial \sigma}{\partial t} &= \frac{\partial^2 \lambda_3}{\partial s^2} + \frac{\partial(\lambda_2\sigma)}{\partial s} - \frac{\partial(\lambda_4\tau)}{\partial s} - \tau\left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right) - \kappa\psi_1 - \psi_3.
\end{aligned}$$

Proof. Noting that

$$\frac{\partial}{\partial t}\frac{\partial T}{\partial s} = \frac{\partial}{\partial t}(\kappa\delta + \sigma N - B) = \frac{\partial \kappa}{\partial t}\delta + \kappa\frac{\partial \delta}{\partial t} + \frac{\partial \sigma}{\partial t}N + \sigma\frac{\partial N}{\partial t} - \frac{\partial B}{\partial t}.$$

By using (3.2), one can write

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial T}{\partial s} &= (\frac{\partial \kappa}{\partial t} + \kappa\psi_2 + \sigma\psi_3)\delta + (\kappa\psi_1 + \frac{\partial \sigma}{\partial t} + \psi_3)N + (-\sigma\psi_1 + \psi_2)\,B\\
&+ \left(-\kappa(\frac{\partial \lambda_4}{\partial s} - \lambda_2) - \sigma(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau) + (\frac{\partial \lambda_1}{\partial s} + \lambda_2\kappa + \lambda_3\tau)\right)T. \qquad (3.4)
\end{aligned}$$

Also, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial s}\left(\left(\frac{\partial \lambda_1}{\partial s} + \lambda_2\kappa + \lambda_3\tau\right)\delta + \left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right)N + \left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right)B\right)\\
&= \left(\frac{\partial^2 \lambda_1}{\partial s^2} + \frac{\partial(\lambda_2\kappa)}{\partial s} + \frac{\partial(\lambda_3\tau)}{\partial s}\right)\delta + \left(\frac{\partial \lambda_1}{\partial s} + \lambda_2\kappa + \lambda_3\tau\right)\delta' +\\
&\left(\frac{\partial^2 \lambda_3}{\partial s^2} + \frac{\partial(\lambda_2\sigma)}{\partial s} - \frac{\partial(\lambda_4\tau)}{\partial s}\right)N + \left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right)N' +\\
&\left(\frac{\partial^2 \lambda_4}{\partial s^2} - \frac{\partial \lambda_2}{\partial s}\right)B + \left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right)B'.
\end{aligned}$$

In the light of (2.1), we obtain

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial T}{\partial t} &= \left(\frac{\partial^2 \lambda_1}{\partial s^2} + \frac{\partial(\lambda_2\kappa)}{\partial s} + \frac{\partial(\lambda_3\tau)}{\partial s} + \tau\left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right)\right)\delta\\
&+ \left(\frac{\partial \lambda_1}{\partial s} + \lambda_2\kappa + \lambda_3\tau - \sigma\left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right) - \kappa\left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right)\right)T \qquad (3.5)\\
&+ \left(\frac{\partial^2 \lambda_3}{\partial s^2} + \frac{\partial(\lambda_2\sigma)}{\partial s} - \frac{\partial(\lambda_4\tau)}{\partial s} - \tau\left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right)\right)N + \left(\frac{\partial^2 \lambda_4}{\partial s^2} - \frac{\partial \lambda_2}{\partial s}\right)B.
\end{aligned}$$

Since $\frac{\partial}{\partial s}\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\frac{\partial T}{\partial s}$, so from (3.4) and (3.5), we get

$$\begin{aligned}
\frac{\partial \kappa}{\partial t} &= \frac{\partial^2 \lambda_1}{\partial s^2} + \frac{\partial(\lambda_2\kappa)}{\partial s} + \frac{\partial(\lambda_3\tau)}{\partial s} + \tau\left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right) - \kappa\psi_2 - \sigma\psi_3,\\
\frac{\partial \sigma}{\partial t} &= \frac{\partial^2 \lambda_3}{\partial s^2} + \frac{\partial(\lambda_2\sigma)}{\partial s} - \frac{\partial(\lambda_4\tau)}{\partial s} - \tau\left(\frac{\partial \lambda_4}{\partial s} - \lambda_2\right) - \kappa\psi_1 - \psi_3.
\end{aligned}$$

Similar to the above procedure that we have considered to get $\frac{\partial \kappa}{\partial t}$ and $\frac{\partial \sigma}{\partial t}$ and using $\frac{\partial}{\partial s}\frac{\partial B}{\partial t} = \frac{\partial}{\partial t}\frac{\partial B}{\partial s}$, we can obtain

$$\frac{\partial \tau}{\partial t} = \sigma\left(\frac{\partial \lambda_1}{\partial s} + \lambda_2\kappa + \lambda_3\tau\right) - \kappa\left(\frac{\partial \lambda_3}{\partial s} + \lambda_2\sigma - \lambda_4\tau\right) - \psi_2\tau + \frac{\partial \psi_3}{\partial s}.$$

Thus the theorem is proved. □

4 Inextensible flow of a lightlike curve in Q^3

After studying the inextensible flow of spacelike curves, we have been able to study the null curves, which resulted in the study of the following results.

Theorem 4.1. *Let* $\{\gamma, e_1, e_2, e_3\}$ *be a natural Frenet frame of a null curve and* $\frac{\partial\Gamma}{\partial t} = \mu_1\gamma + \mu_2 e_1 + \mu_3 e_2 + \mu_4 e_3$ *be a smooth flow of* Γ *in* $Q^3 \subset E_1^4$*, then we have*

$$\frac{\partial v}{\partial t} = \left(\frac{\partial \mu_3}{\partial u} + v\mu_3 h - v\mu_4\kappa_1\right).$$

Lemma 4.2. *Let* $\frac{\partial\Gamma}{\partial t} = \mu_1\gamma + \mu_2 e_1 + \mu_3 e_2 + \mu_4 e_3$ *be a smooth flow of* Γ *in* Q^3*, then the flow is inextensible if and only if*

$$\frac{\partial \mu_3}{\partial s} = \mu_4\kappa_1 - \mu_3 h.$$

Lemma 4.3. *Let* $\{\gamma, e_1, e_2, e_3\}$ *be a natural Frenet frame of a null curve in* $Q^3 \subset E_1^4$*, then we have the following differentiations*

$$\begin{aligned}
\frac{\partial \gamma}{\partial t} &= \Psi_2\gamma + \Psi_1 e_1 - \left(\frac{\partial \mu_4}{\partial s} - \mu_3\right) e_2,\\
\frac{\partial e_1}{\partial t} &= \left(\frac{\partial \mu_1}{\partial s} + \mu_2\kappa_1 + \mu_3\kappa_2\right)\gamma + \left(\frac{\partial \mu_2}{\partial s} + \mu_1 + \mu_2 h - \mu_4\kappa_2\right) e_1 + \left(\frac{\partial \mu_4}{\partial s} - \mu_3\right) e_3,\\
\frac{\partial e_2}{\partial t} &= \Psi_3\gamma - \left(\frac{\partial \mu_2}{\partial s} + \mu_1 + \mu_2 h - \mu_4\kappa_2\right) e_2 - \Psi_1 e_3,\\
\frac{\partial e_3}{\partial t} &= -\Psi_3 e_1 - \left(\frac{\partial \mu_1}{\partial s} + \mu_2\kappa_1 + \mu_3\kappa_2\right) e_2 - \Psi_2 e_3.
\end{aligned}$$

where

$$\Psi_1 = \langle\frac{\partial \gamma}{\partial t}, e_2\rangle,\quad \Psi_2 = \langle\frac{\partial \gamma}{\partial t}, e_3\rangle \text{ and } \Psi_3 = \langle\frac{\partial e_2}{\partial t}, e_3\rangle.$$

Theorem 4.4. *Suppose that* $\frac{\partial\Gamma}{\partial t}$ *is inextensible, then we obtain the following system of partial differential equations:*

$$\frac{\partial \Psi_1}{\partial s} = \frac{\partial \mu_2}{\partial s} + \mu_1 + \mu_2 h - \mu_4\kappa_2 - \Psi_2 - \Psi_1 h,\ \frac{\partial \Psi_2}{\partial s} = \frac{\partial \mu_1}{\partial s} + \mu_2\kappa_1 + \kappa_2\frac{\partial \mu_4}{\partial s} - \kappa_1\Psi_1,$$

Theorem 4.5. *Let* $\frac{\partial\Gamma}{\partial t}$ *be an inextensible flow in* $Q^3 \subset E_1^4$*, then we get the following differentiations of curvatures*

$$\begin{aligned}
\frac{\partial \kappa_1}{\partial t} &= \frac{\partial^2 \mu_1}{\partial s^2} + \frac{\partial(\mu_2\kappa_1)}{\partial s} + \frac{\partial(\mu_3\kappa_2)}{\partial s} + \kappa_1\left(\frac{\partial \mu_2}{\partial s} + \mu_1 - \mu_4\kappa_2\right) - h\frac{\partial \mu_1}{\partial s} + h\mu_3\kappa_2 - \kappa_1\Psi_2,\\
\frac{\partial \kappa_2}{\partial t} &= \frac{\partial \Psi_3}{\partial s} + \Psi_3 h - \Psi_2\kappa_2 - \kappa_2\left(\frac{\partial \mu_2}{\partial s} + \mu_1 + \mu_2 h - \mu_4\kappa_2\right),\\
\frac{\partial h}{\partial t} &= \frac{\partial^2 \mu_2}{\partial s^2} + \frac{\partial \mu_1}{\partial s} + \frac{\partial(\mu_2 h)}{\partial s} - \frac{\partial(\mu_4\kappa_2)}{\partial s} + \frac{\partial \mu_1}{\partial s} + \mu_2\kappa_1 - \kappa_2\frac{\partial \mu_4}{\partial s} - \kappa_1\Psi_1.
\end{aligned}$$

Acknowledgment

This work was supported by the Funding Agency xpto.

References

Gage, M. 1986. On an area-preserving evolution equation for plane curves, *Contemp. Math* **51**: 51–62.

Gürbüz, N. 2009. Inextensible flows of spacelike, timelike and null curves, *Int. J. Contemp. Math. Sciences* **4**(32): 1599–1604.

Latifi, D. and Razavi, A. 2008. Inextensible flows of curves in minkowskian space, *Advanced Studies in Theoretical Physics* **2**(16): 761–768.

Liu, H. 2004. Curves in the lightlike cone, *Contributions to Algebra and Geometry* **45**(1): 291–303.

Liu, H. and Meng, Q. 2011. Representation formulas of curves in a two-and three-dimensional lightlike cone, *Results in Mathematics* **59**(3-4): 437.

O'neill, B. 1983. *Semi-Riemannian geometry with applications to relativity*, Vol. 103, Academic press.

Sapiro, G. 2001. Geometric partial differential equations and image analysis, cambridgeuniv, *Press, Cambridge, UK* .

Sun, J. and Pei, D. 2015. Some new properties of null curves on 3-null cone and unit semi-euclidean 3-spheres, *J. Nonlinear Sci. Appl* **8**: 275–284.

Yeneroğlu, M. et al. 2010. Inextensible curves in the galilean space, *International Journal of Physical Sciences* **5**(9): 1424–1427.