wave equation for **H** can be rewritten in the form

$$\frac{\partial^2 \mathbf{H}}{\partial x^2}+\frac{\partial^2 \mathbf{H}}{\partial y^2}+\frac{\partial^2 \mathbf{H}}{\partial z^2}=\gamma^2 \mathbf{H}$$

Of particular interest are solutions (*plane waves*) that depend on only one spatial coordinate, say z. Then the equation becomes

$$\frac{d^2 \mathbf{H}}{dz^2}=\gamma^2 \mathbf{H}$$

which, for an assumed time dependence $e^{j\omega t}$, is the vector analog of the one-dimensional scalar wave equation. Solutions are as above, in terms of the propagation constant γ.

$$\mathbf{H}(z, t)=H_0 e^{\pm \gamma z} e^{j\omega t} \mathbf{a}_H$$

The corresponding solutions for the electric field are

$$\mathbf{E}(z, t)=E_0 e^{\pm \gamma z} e^{j\omega t} \mathbf{a}_E$$

The fixed unit vectors $\mathbf{a}_H$ and $\mathbf{a}_E$ are orthogonal and neither field has a component in the direction of propagation. This being the case, one can rotate the axes to put one of the fields, say **E**, along the x axis. Then from Maxwell's equation (*2*) it follows that **H** will lie along the $\pm y$ axis for propagation in the $\pm z$ direction.

EXAMPLE 1. Given the field $\mathbf{E}=E_0 e^{-\gamma z}\mathbf{a}_E$ (time dependence suppressed), show that **E** can have no component in the propagation direction, $+\mathbf{a}_z$.

The cartesian components of $\mathbf{a}_E$ are found by projection:

$$\mathbf{E}=E_0 e^{-\gamma z}[(\mathbf{a}_E \cdot \mathbf{a}_x)\mathbf{a}_x+(\mathbf{a}_E \cdot \mathbf{a}_y)\mathbf{a}_y+(\mathbf{a}_E \cdot \mathbf{a}_z)\mathbf{a}_z]$$

From $\nabla \cdot \mathbf{E}=0$,

$$\frac{\partial}{\partial z} E_0 e^{-\gamma z}(\mathbf{a}_E \cdot \mathbf{a}_z)=0$$

which can hold only if $\mathbf{a}_E \cdot \mathbf{a}_z=0$. Consequently, E has no component in $\mathbf{a}_z$.

The plane wave solutions obtained above depend on the properties μ, ϵ, and σ of the medium, because these properties are involved in the propagation constant γ.

14.4 SOLUTIONS FOR PARTIALLY CONDUCTING MEDIA

For a region in which there is some conductivity but not much (e.g., moist earth, seawater), the solution to the wave equation in **E** is taken to be

$$\mathbf{E}=E_0 e^{-\gamma z}\mathbf{a}_x$$

Then, from (*2*) of Section 14.2,

$$\mathbf{H}=\sqrt{\frac{\sigma+j\omega\epsilon}{j\omega\mu}}\, E_0 e^{-\gamma z}\mathbf{a}_y$$

The ratio E/H is characteristic of the medium (it is also frequency-dependent). More specifically for waves $\mathbf{E}=E_x\mathbf{a}_x$, $\mathbf{H}=H_y\mathbf{a}_y$ which propagate in the $+z$ direction, the *intrinsic impedance*, η, of the medium is defined by

$$\eta=\frac{E_x}{H_y}$$

Thus

$$\eta=\sqrt{\frac{j\omega\mu}{\sigma+j\omega\epsilon}}$$

where the correct square root may be written in polar form, $|\eta| \underline{/\theta}$, with

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}} \qquad \tan 2\theta = \frac{\sigma}{\omega\epsilon} \qquad \text{and} \qquad 0° < \theta < 45°$$

(If the wave propagates in the $-z$ direction, $E_x/H_y = -\eta$. In effect, γ is replaced by $-\gamma$ and the other square root used.)

Inserting the time factor $e^{j\omega t}$ and writing $\gamma = \alpha + j\beta$ results in the following equations for the fields in a partially conducting region:

$$\mathbf{E}(z, t) = E_0 e^{-\alpha z} e^{j(\omega t - \beta z)} \mathbf{a}_x$$

$$\mathbf{H}(z, t) = \frac{E_0}{|\eta|} e^{-\alpha z} e^{j(\omega t - \beta z - \theta)} \mathbf{a}_y$$

The factor $e^{-\alpha z}$ attenuates the magnitudes of both **E** and **H** as they propagate in the $+z$ direction. The expression for α, (5) of Section 14.2, shows that there will be some attenuation unless the conductivity σ is zero, which would be the case only for perfect dielectrics or free space. Likewise, the phase difference θ between $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ vanishes only when σ is zero.

The velocity of propagation and the wavelength are given by

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right)}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\frac{\mu\epsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right)}}$$

If the propagation velocity is known, $\lambda f = u$ may be used to determine the wavelength λ. The term $(\sigma/\omega\epsilon)^2$ has the effect of reducing both the velocity and the wavelength from what they would be in either free space or perfect dielectrics, where $\sigma = 0$. Observe that the medium is *dispersive*: waves with different frequencies ω have different velocities u.

14.5 SOLUTIONS FOR PERFECT DIELECTRICS

For a perfect dielectric, $\sigma = 0$, and so

$$\alpha = 0 \qquad \beta = \omega\sqrt{\mu\epsilon} \qquad \eta = \sqrt{\frac{\mu}{\epsilon}} \underline{/0°}$$

Since $\alpha = 0$, there is no attenuation of the **E** and **H** waves. The zero angle on η results in **H** being in time phase with **E** at each fixed location. Assuming **E** in $\mathbf{a}_x$ and propagation in $\mathbf{a}_z$, the field equations may be obtained as limits of those in Section 14.4:

$$\mathbf{E}(z, t) = E_0 e^{j(\omega t - \beta z)} \mathbf{a}_x$$

$$\mathbf{H}(z, t) = \frac{E_0}{\eta} e^{j(\omega t - \beta z)} \mathbf{a}_y$$

The velocity and the wavelength are

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}} \qquad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\epsilon}}$$