Chapter 4

Laplace Transforms

DEFINITION OF A LAPLACE TRANSFORM

The Laplace transform of a function $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{1}$$

and is said to exist or not according as the integral in (*1*) exists [converges] or does not exist [diverges]. In this chapter we assume that s is real. Later [Chapter 14] we shall find it convenient to take s as complex.

Often in practice there will be a real number s_0 such that the integral (*1*) exists for $s > s_0$ and does not exist for $s \leqq s_0$. The set of values $s > s_0$ for which (*1*) exists is called the *range of convergence* or *existence* of $\mathcal{L}\{f(t)\}$. It may happen, however, that (*1*) does not exist for any value of s [see Problem 4.50].

The symbol $\mathcal{L}$ in (*1*) is called the *Laplace transform operator.* We can show that $\mathcal{L}$ is a linear operator, i.e.

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} \tag{2}$$

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

In the following table we give Laplace transforms of some special elementary functions together with the range of existence or convergence. Often, however, we shall omit this range of existence since in most instances it can easily be supplied when needed.

	$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
1.	1	$\dfrac{1}{s}$ $s > 0$
2.	t^n $n = 1, 2, 3, \ldots$	$\dfrac{n!}{s^{n+1}}$ $s > 0$
3.	t^p $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$ $s > 0$
4.	e^{at}	$\dfrac{1}{s-a}$ $s > a$
5.	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$ $s > 0$
6.	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$ $s > 0$
7.	$\cosh at$	$\dfrac{a}{s^2-a^2}$ $s > \lvert a \rvert$
8.	$\sinh at$	$\dfrac{s}{s^2-a^2}$ $s > \lvert a \rvert$

In entry 3, $\Gamma(p+1)$ is the *gamma function* defined by

$$\Gamma(p+1) \quad = \quad \int_0^\infty x^p e^{-x}\,dx \qquad p > -1 \tag{3}$$

A study of this function is provided in Chapter 9. For our present purposes, however, we need only the following properties:

$$\Gamma(p+1) \;=\; p\Gamma(p), \quad \Gamma(\tfrac{1}{2}) \;=\; \sqrt{\pi}, \quad \Gamma(1) \;=\; 1 \tag{4}$$

The first is called a *recursion formula* for the gamma function. Note that if p is any positive integer n then $\Gamma(n+1) = n!$, thus explaining the relationship of entries 2 and 3 of the table.

SUFFICIENT CONDITIONS FOR EXISTENCE OF LAPLACE TRANSFORMS

In order to be able to state sufficient conditions on $f(t)$ under which we can guarantee the existence of $\mathcal{L}\{f(t)\}$, we introduce the concepts of *piecewise continuity* and *exponential order* as follows.

1. **Piecewise continuity.** A function $f(t)$ is said to be *piecewise continuous* in an interval if (i) the interval can be divided into a finite number of subintervals in each of which $f(t)$ is continuous and (ii) the limits of $f(t)$ as t approaches the endpoints of each subinterval are finite. Another way of stating this is to say that a piecewise continuous function is one that has only a finite number of finite discontinuities. An example of a piecewise continuous function is shown in Fig. 4-1.

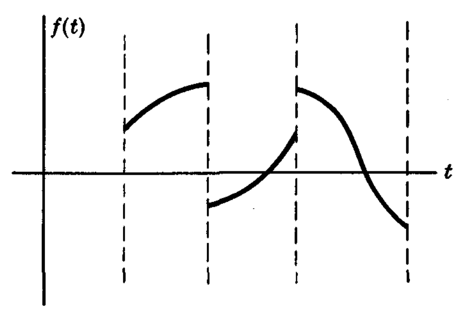

Fig. 4-1

2. **Exponential order.** A function $f(t)$ is said to be of *exponential order* for $t > T$ if we can find constants M and α such that $|f(t)| \leqq Me^{\alpha t}$ for $t > T$.

Using these we have the following theorem,

Theorem 4-1. If $f(t)$ is piecewise continuous in every finite interval $0 \leqq t \leqq T$ and is of exponential order for $t > T$, then $\mathcal{L}\{f(t)\}$ exists for $s > \alpha$.

It should be emphasized that these conditions are only sufficient [and not necessary], i.e. if the conditions are not satisfied $\mathcal{L}\{f(t)\}$ may still exist. For example, $\mathcal{L}\{t^{-1/2}\}$ exists even though $t^{-1/2}$ is not piecewise continuous in $0 \leqq t \leqq T$.

An interesting theorem which is related to Theorem 4-1 is the following

Theorem 4-2. If $f(t)$ satisfies the conditions of Theorem 4-1, then

$$\lim_{s\to\infty} \mathcal{L}\{f(t)\} \;=\; \lim_{s\to\infty} F(s) \;=\; 0$$

It follows that if $\lim_{s\to\infty} F(s) \neq 0$, then $f(t)$ cannot satisfy the conditions of Theorem 4-1.

INVERSE LAPLACE TRANSFORMS

If $\mathcal{L}\{f(t)\} = F(s)$, then we call $f(t)$ the *inverse Laplace transform* of $F(s)$ and write $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Example 1. Since $\mathcal{L}\{t\} = \dfrac{1}{s^2}$, we have $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$.

While it is clear that whenever a Laplace transform exists it is unique, the same is not true for an inverse Laplace transform.

Example 2. If $f(t) = \begin{cases} t & t \neq 2 \\ 10 & t = 2 \end{cases}$ we can show that $\mathcal{L}\{f(t)\} = 1/s^2$. However, this function $f(t)$ differs from that of Example 1 at $t = 2$ although both have the same Laplace transform. It follows that $\mathcal{L}^{-1}(1/s^2)$ can represent two (or more) different functions.

We can show that if two functions have the same Laplace transform, then they cannot differ from each other on any interval of positive length no matter how small. This is sometimes called *Lerch's theorem*. The theorem implies that if two functions have the same Laplace transform, then they are for all practical purposes the same and so in practice we can take the inverse Laplace transform as *essentially* unique. In particular if two continuous functions have the same Laplace transform, they must be identical.

The symbol $\mathcal{L}^{-1}$ is called the *inverse Laplace transform* operator and is a linear operator, i.e.

$$\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s)\} = c_1f_1(t) + c_2f_2(t)$$

LAPLACE TRANSFORMS OF DERIVATIVES

We shall find that Laplace transforms provide useful means for solving linear differential equations. For this reason it will be necessary for us to find Laplace transforms of derivatives. The following theorems are fundamental.

Theorem 4-3. Let $f(t)$ be continuous and have a piecewise continuous derivative $f'(t)$ in every finite interval $0 \leqq t \leqq T$. Suppose also that $f(t)$ is of exponential order for $t > T$. Then

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

This can be extended as follows.

Theorem 4-4. Let $f(t)$ be such that $f^{(n-1)}(t)$ is continuous and $f^{(n)}(t)$ piecewise continuous in every finite interval $0 \leqq t \leqq T$. Suppose also that $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are of exponential order for $t > T$. Then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

THE UNIT STEP FUNCTION

The *unit step function*, also called *Heaviside's unit step function*, is defined as

$$\mathcal{U}(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

and is shown graphically in Fig. 4-2.

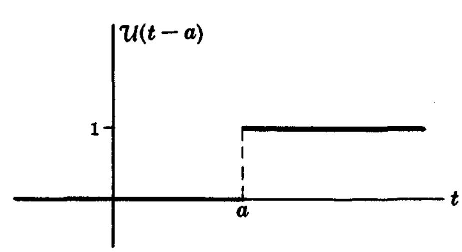

Fig. 4-2

It is possible to express various discontinuous functions in terms of the unit step function.

We can show [Problem 4.17] that the Laplace transform of the unit step function is

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s} \qquad s > 0$$

and similarly we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = \mathcal{U}(t-a)$$

SOME SPECIAL THEOREMS ON LAPLACE TRANSFORMS

Because of the relationship between Laplace transforms and inverse Laplace transforms, any theorem involving Laplace transforms will have a corresponding theorem involving inverse Laplace transforms. In the following we shall consider some of the important results involving Laplace transforms and corresponding inverse Laplace transforms. In all cases we assume that $f(t)$ satisfies the conditions of Theorem 4-1.

***Theorem 4-5* [First translation theorem].** If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

Similarly if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

***Theorem 4-6* [Second translation theorem].** If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{\mathcal{U}(t-a)\,f(t-a)\} = e^{-as}F(s)$$

Similarly if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = \mathcal{U}(t-a)\,f(t-a)$$

Theorem 4-7. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Similarly if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\left\{F\left(\frac{s}{a}\right)\right\} = af(at)$$

Theorem 4-8. If $\mathcal{L}\{f(t)\} = F(s)$ then if $n = 1, 2, 3, \ldots$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^nF}{ds^n} = (-1)^n F^{(n)}(s)$$

Similarly if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t)$$

***Theorem 4-9* [Periodic functions].** If $f(t)$ has period $P > 0$, i.e. if $f(t+P) = f(t)$, then

$$\mathcal{L}\{f(t)\} = \frac{\int_0^P e^{-st}f(t)\,dt}{1 - e^{-sP}}$$

***Theorem 4-10* [Integration].** If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s}$$

Similarly if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$$

Theorem 4-11. If $\lim_{t\to 0} \frac{f(t)}{t}$ exists and $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

Theorem 4-12 **[Convolution theorem].** If $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$, then

$$\mathcal{L}\left\{\int_0^t f(u)\, g(t-u)\, du\right\} = F(s)\, G(s)$$

Similarly if $\mathcal{L}^{-1}\{F(s)\} = f(t)$, $\mathcal{L}^{-1}\{G(s)\} = g(t)$, then

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = \int_0^t f(u)\, g(t-u)\, du$$

We call the above integral the *convolution* of f and g and write

$$f*g = \int_0^t f(u)\, g(t-u)\, du$$

We have the result $f*g = g*f$, i.e. the convolution is commutative. Similarly we can prove that it is associative and distributive [see Problem 4.75].

PARTIAL FRACTIONS

Although the above theorems are often useful in finding inverse Laplace transforms, perhaps the most important single elementary method for our purposes is the method of *partial fractions*. This is because in many problems which we shall encounter it will be necessary to find the inverse of $P(s)/Q(s)$ where $P(s)$ and $Q(s)$ are polynomials and the degree of $Q(s)$ is larger than that of $P(s)$. For illustrations of the method see Problems 4.39-4.41.

SOLUTIONS OF DIFFERENTIAL EQUATIONS BY LAPLACE TRANSFORMS

The method of Laplace transforms is particularly useful for solving linear differential equations with constant coefficients and associated initial conditions. To accomplish this we take the Laplace transform of the given differential equation [or equations in the case of a system], making use of the initial conditions. This leads to an algebraic equation [or system of algebraic equations] in the Laplace transform of the required solution. By solving for this Laplace transform and then taking the inverse, the required solution is obtained. For illustrations see Problems 4.42-4.44.

APPLICATIONS TO PHYSICAL PROBLEMS

Since formulation of many physical problems leads to linear differential equations with initial conditions, the Laplace transform method is particularly suited for obtaining their solutions. For applications to various fields see Problems 4.45-4.47.

LAPLACE INVERSION FORMULAS

There exists a direct method for finding inverse Laplace transforms, called the *complex inversion formula*. This makes use of the theory of complex variables and is considered in Chapter 14.

Solved Problems

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

4.1. Prove that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ if $s > a$.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{e^{-(s-a)t}}{s-a}\bigg|_0^\infty$$

$$= \frac{1}{s-a} \text{ provided } s - a > 0, \text{ i.e. } s > a$$

4.2. (a) Prove that $\mathcal{L}\{t^p\} = \dfrac{\Gamma(p+1)}{s^{p+1}}$ if $s > 0$ and $p > -1$.

(b) Show that if $p = n$, a positive integer, then $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ where $s > 0$.

(a) $\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p\,dt$. Let $st = u$ and note that in order for the integral to converge we must have $s > 0$. Then the integral equals

$$\frac{1}{s^{p+1}}\int_0^\infty u^p e^{-u}\,du = \frac{\Gamma(p+1)}{s^{p+1}}$$

The restriction $p > -1$ occurs because the integral defining the gamma function converges if and only if $p > -1$.

(b) Integrating by parts, we have

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\,dx = (x^p)(-e^{-x})\bigg|_0^\infty - \int_0^\infty (px^{p-1})(-e^{-x})\,dx$$

$$= p\int_0^\infty x^{p-1}e^{-x}\,dx = p\Gamma(p)$$

i.e.

$$\Gamma(p+1) = p\Gamma(p)$$

If $p = n$, then

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) = n(n-1)\cdots 1\Gamma(1)$$

But $\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$. Thus $\Gamma(n+1) = n!$ and so from (a), $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$.

4.3. Prove that $\mathcal{L}$ is a linear operator.

We must show that if c_1, c_2 are any constants and $f_1(t), f_2(t)$ any functions whose Laplace transforms exist, then

$$\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

We have

$$\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = \int_0^\infty e^{-st}\{c_1f_1(t) + c_2f_2(t)\}\,dt$$

$$= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt$$

$$= c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

and so $\mathcal{L}$ is a linear operator.

4.4. Prove that (a) $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2+\omega^2}$, (b) $\mathcal{L}\{\sin \omega t\} = \dfrac{\omega}{s^2+\omega^2}$.

Method 1. We have if $s > 0$,

$$\int_0^\infty e^{-st}e^{i\omega t}\,dt = \int_0^\infty e^{-(s-i\omega)t}\,dt = \left.\frac{e^{-(s-i\omega)t}}{s-i\omega}\right|_0^\infty = \frac{1}{s-i\omega}$$

Then taking real and imaginary parts, we have

$$\int_0^\infty e^{-st}(\cos \omega t + i \sin \omega t)\,dt = \frac{s+i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

or

$$\int_0^\infty e^{-st}\cos \omega t\,dt = \frac{s}{s^2+\omega^2}, \qquad \int_0^\infty e^{-st}\sin \omega t\,dt = \frac{\omega}{s^2+\omega^2}$$

Method 2. By direct integration,

$$\int_0^\infty e^{-st}\cos \omega t\,dt = \left.\frac{e^{-st}(\omega \sin \omega t - s \cos \omega t)}{s^2+\omega^2}\right|_0^\infty = \frac{s}{s^2+\omega^2}$$

$$\int_0^\infty e^{-st}\sin \omega t\,dt = \left.\frac{-e^{-st}(s \sin \omega t + \omega \cos \omega t)}{s^2+\omega^2}\right|_0^\infty = \frac{\omega}{s^2+\omega^2}$$

4.5. Find the Laplace transforms of each of the following:
(a) $3e^{-4t}$, (b) $2t^2$, (c) $4\cos 5t$, (d) $\sin \pi t$, (e) $-3/\sqrt{t}$.

(a) $\mathcal{L}\{3e^{-4t}\} = 3\mathcal{L}\{e^{-4t}\} = \dfrac{3}{s-(-4)} = \dfrac{3}{s+4}, \quad s > -4$

(b) $\mathcal{L}\{2t^2\} = 2\mathcal{L}\{t^2\} = \dfrac{2\Gamma(3)}{s^3} = \dfrac{2\cdot 2!}{s^3} = \dfrac{4}{s^3}, \quad s > 0$

(c) $\mathcal{L}\{4\cos 5t\} = 4\mathcal{L}\{\cos 5t\} = 4\cdot\dfrac{s}{s^2+25} = \dfrac{4s}{s^2+25}, \quad s > 0$

(d) $\mathcal{L}\{\sin \pi t\} = \dfrac{\pi}{s^2+\pi^2}, \quad s > 0$

(e) $\mathcal{L}\left\{-\dfrac{3}{\sqrt{t}}\right\} = -3\mathcal{L}\{t^{-1/2}\} = -\dfrac{3\Gamma(1/2)}{s^{1/2}} = -\dfrac{3\sqrt{\pi}}{\sqrt{s}} = -3\sqrt{\dfrac{\pi}{s}}, \quad s > 0$

4.6. Find the Laplace transforms of each of the following:
(a) $3t^4 - 2t^{3/2} + 6$, (b) $5\sin 2t - 3\cos 2t$, (c) $3\sqrt[3]{t} + 4e^{2t}$, (d) $1/t^2$.

(a)
$$\begin{aligned}\mathcal{L}\{3t^4 - 2t^{3/2} + 6\} &= 3\mathcal{L}\{t^4\} - 2\mathcal{L}\{t^{3/2}\} + 6\mathcal{L}\{1\} \\ &= \frac{3\Gamma(5)}{s^5} - \frac{2\Gamma(5/2)}{s^{5/2}} + \frac{6}{s} \\ &= \frac{3\cdot 4!}{s^5} - \frac{2\cdot(3/2)(1/2)\Gamma(1/2)}{s^{5/2}} + \frac{6}{s} \\ &= \frac{72}{s^5} - \frac{3\sqrt{\pi}}{2s^{5/2}} + \frac{6}{s}\end{aligned}$$

(b)
$$\begin{aligned}\mathcal{L}\{5\sin 2t - 3\cos 2t\} &= 5\mathcal{L}\{\sin 2t\} - 3\mathcal{L}\{\cos 2t\} \\ &= \frac{5\cdot 2}{s^2+4} - \frac{3\cdot s}{s^2+4} = \frac{10-3s}{s^2+4}\end{aligned}$$

(c)
$$\begin{aligned}\mathcal{L}\{3\sqrt[3]{t} + 4e^{2t}\} &= 3\mathcal{L}\{t^{1/3}\} + 4\mathcal{L}\{e^{2t}\} = \frac{3\Gamma(4/3)}{s^{4/3}} + \frac{4}{s-2} \\ &= \frac{3(1/3)\Gamma(1/3)}{s^{4/3}} + \frac{4}{s-2} = \frac{\Gamma(1/3)}{s^{4/3}} + \frac{4}{s-2}\end{aligned}$$

(d) $\mathcal{L}\left\{\dfrac{1}{t^2}\right\} = \displaystyle\int_0^\infty \frac{e^{-st}}{t^2}\,dt$. Since this integral does not converge, the Laplace transform does not exist.

In parts (a), (b), (c) we have omitted the range of existence which can easily be supplied.

4.7. If $f(t) = \begin{cases} 3 & 0 < t < 2 \\ -1 & 2 < t < 4 \\ 0 & t \geqq 4 \end{cases}$ find $\mathcal{L}\{f(t)\}$.

We have

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^2 e^{-st} f(t)\,dt + \int_2^4 e^{-st} f(t)\,dt + \int_4^\infty e^{-st} f(t)\,dt \\ &= \int_0^2 e^{-st}(3)\,dt + \int_2^4 e^{-st}(-1)\,dt + \int_4^\infty e^{-st}(0)\,dt \\ &= 3\left(\frac{e^{-st}}{-s}\right)\Bigg|_0^2 - \left(\frac{e^{-st}}{-s}\right)\Bigg|_2^4 + 0 \\ &= \frac{3(1-e^{-2s})}{s} + \frac{e^{-4s}-e^{-2s}}{s} = \frac{3-4e^{-2s}+e^{-4s}}{s} \end{aligned}$$

4.8. Find $\mathcal{L}\{\sin t \cos t\}$.

We have $\sin 2t = 2 \sin t \cos t$ so that $\sin t \cos t = \frac{1}{2} \sin 2t$. Thus

$$\mathcal{L}\{\sin t \cos t\} = \frac{1}{2}\mathcal{L}\{\sin 2t\} = \frac{1}{2} \cdot \frac{2}{s^2+4} = \frac{1}{s^2+4}$$

EXISTENCE OF LAPLACE TRANSFORMS

4.9. Prove (*a*) Theorem 4-1, and (*b*) Theorem 4-2, page 99.

(*a*) We have

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

Now since $f(t)$ is piecewise continuous for $0 \leqq t \leqq T$ so also is $e^{-st} f(t)$, and thus the first integral on the right exists.

To show that the second integral on the right also exists, we use the fact that $|f(t)| \leqq Me^{\alpha t}$ so that

$$\begin{aligned} \left|\int_T^\infty e^{-st} f(t)\,dt\right| &\leqq \int_T^\infty e^{-st}|f(t)|\,dt \leqq \int_T^\infty e^{-st} M e^{\alpha t}\,dt \\ &\leqq M\int_0^\infty e^{-(s-\alpha)}\,dt = \frac{M}{s-\alpha} \end{aligned} \tag{1}$$

for $s > \alpha$ and the required result is proved.

(*b*) We have as in part (*a*),

$$|F(s)| = |\mathcal{L}\{f(t)\}| \leqq \int_0^T e^{-st}|f(t)|\,dt + \int_T^\infty e^{-st}|f(t)|\,dt$$

Now since $f(t)$ is piecewise continuous for $0 \leqq t \leqq T$, it is bounded, i.e. $|f(t)| \leqq K$ for some constant K. Using this and the result (*1*), we have

$$|F(s)| \leqq \int_0^T e^{-st} K\,dt + \frac{M}{s-\alpha} \leqq \int_0^\infty e^{-st} K\,dt + \frac{M}{s-\alpha} = \frac{K+M}{s-\alpha}$$

Taking the limit as $s \to \infty$, it follows that $\lim_{s\to\infty} F(s) = 0$ as required.

4.10. Prove that (*a*) $\mathcal{L}\left\{\frac{e^{2t}}{t+4}\right\}$ exists, (*b*) $\lim_{s\to\infty} \mathcal{L}\left\{\frac{e^{2t}}{t+4}\right\} = 0$.

(*a*) In every finite interval $e^{2t}/(t+4)$ is continuous [and thus certainly piecewise continuous]. Also for all $t \geqq 0$,

$$\frac{e^{2t}}{t+4} < \frac{e^{2t}}{4}$$

so that $e^{2t}/(t+4)$ is of exponential order. Thus by Problem 4.9(*a*) the Laplace transform exists.

(*b*) This follows at once from Problem 4.9(*b*) and the results of (*a*).

SOME ELEMENTARY INVERSE LAPLACE TRANSFORMS

4.11. Prove that $\mathcal{L}^{-1}$ is a linear operator.

We have, since $\mathcal{L}$ is a linear operator [Problem 4.3],

$$\mathcal{L}\{c_1f_1(t) + c_2f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} = c_1F_1(s) + c_2F_2(s)$$

Thus by definition

$$\begin{aligned}\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s)\} &= c_1f_1(t) + c_2f_2(t)\\ &= c_1\mathcal{L}^{-1}\{F_1(s)\} + c_2\mathcal{L}^{-1}\{F_2(s)\}\end{aligned}$$

which shows that $\mathcal{L}^{-1}$ is a linear operator.

4.12. Find (a) $\mathcal{L}^{-1}\left\{\dfrac{5}{s+2}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{4s-3}{s^2+4}\right\}$, (c) $\mathcal{L}^{-1}\left(\dfrac{2s-5}{s^2}\right)$, (d) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^k}\right\}$, $k > 0$.

(a) $$\mathcal{L}^{-1}\left\{\frac{5}{s+2}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = 5e^{-2t}$$

(b) $$\mathcal{L}^{-1}\left\{\frac{4s-3}{s^2+4}\right\} = 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = 4\cos 2t - \frac{3}{2}\sin 2t$$

(c) $$\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 2 - 5t$$

(d) Since $\mathcal{L}\{t^p\} = \dfrac{\Gamma(p+1)}{s^{p+1}}$,

$$\mathcal{L}\left\{\frac{t^p}{\Gamma(p+1)}\right\} = \frac{1}{s^{p+1}} \quad \text{or} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^{p+1}}\right\} = \frac{t^p}{\Gamma(p+1)}$$

Then letting $p = k-1$,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^k}\right\} = \frac{t^{k-1}}{\Gamma(k)}$$

4.13. Find (a) $\mathcal{L}^{-1}\left\{\dfrac{4-5s}{s^{3/2}}\right\}$, (b) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+2s}\right\}$.

(a) $$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{4-5s}{s^{3/2}}\right\} &= 4\mathcal{L}^{-1}\left\{\frac{1}{s^{3/2}}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{s^{1/2}}\right\}\\ &= 4\cdot\frac{t^{1/2}}{\Gamma(3/2)} - 5\cdot\frac{t^{-1/2}}{\Gamma(1/2)}\\ &= \frac{8t^{1/2}}{\sqrt{\pi}} - \frac{5t^{-1/2}}{\sqrt{\pi}} = \frac{8t^{1/2} - 5t^{-1/2}}{\sqrt{\pi}}\end{aligned}$$

(b) $$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \tfrac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+2}\right\}\\ &= \tfrac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \tfrac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\\ &= \tfrac{1}{2} - \tfrac{1}{2}e^{-2t} = \tfrac{1}{2}(1-e^{-2t})\end{aligned}$$

LAPLACE TRANSFORMS OF DERIVATIVES

4.14. Prove Theorem 4-3, page 100.

Since $f'(t)$ is piecewise continuous in $0 \leqq t \leqq T$, there exists a finite number of subintervals, say $(0, T_1), (T_1, T_2), \ldots, (T_n, T)$, in each of which $f'(t)$ is continuous and where the limits of $f'(t)$ as t approaches the endpoints of each subinterval are finite. Then

$$\int_0^T e^{-st}f'(t)\,dt = \int_0^{T_1} e^{-st}f'(t)\,dt + \int_{T_1}^{T_2} e^{-st}f'(t)\,dt + \cdots + \int_{T_n}^{T} e^{-st}f'(t)\,dt$$

The right side becomes on integrating by parts,

$$\left[e^{-st}f(t)\Big|_0^{T_1} + s\int_0^{T_1} e^{-st}f(t)\,dt\right] + \left[e^{-st}f(t)\Big|_{T_1}^{T_2} + s\int_{T_1}^{T_2} e^{-st}f(t)\,dt\right]$$
$$+ \cdots + \left[e^{-st}f(t)\Big|_{T_n}^{T} + s\int_{T_n}^{T} e^{-st}f(t)\,dt\right]$$

Since $f(t)$ is continuous, we can thus write

$$\int_0^T e^{-st}f'(t)\,dt = e^{-st}f(T) - f(0) + s\int_0^T e^{-st}f(t)\,dt \qquad (1)$$

Now since $f(t)$ is of exponential order, we have if T is large enough,

$$|e^{-sT}f(T)| \leqq |e^{-sT}Me^{\alpha T}| = Me^{-(s-\alpha)T} \qquad (2)$$

Then taking the limit as $T \to \infty$ in (1), using the fact that $\lim_{T\to\infty} e^{-st}f(T) = 0$, we have as required

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)\,dt = s\int_0^\infty e^{-st}f(t)\,dt - f(0) = s\mathcal{L}\{f(t)\} - f(0)$$

4.15. Prove that if $f'(t)$ is continuous and $f''(t)$ is piecewise continuous in every finite interval $0 \leqq t \leqq T$ and if $f(t)$ and $f'(t)$ are of exponential order for $t > T$, then

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Let $g(t) = f'(t)$. Then $g(t)$ satisfies the conditions of Theorem 4-3 so that

$$\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$$

Thus

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0)\\ &= s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0)\\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)\end{aligned}$$

4.16. Let $f(t) = te^{at}$. (a) Show that $f(t)$ satisfies the equation $f'(t) = af(t) + e^{at}$. (b) Use part (a) to find $\mathcal{L}\{te^{at}\}$.

(a) $f'(t) = t(ae^{at}) + e^{at} = af(t) + e^{at}$

(b) $\mathcal{L}\{f'(t)\} = \mathcal{L}\{af(t) + e^{at}\} = a\mathcal{L}\{f(t)\} + \mathcal{L}\{e^{at}\}$

Thus using Problem 4.14, we have since $f(0) = 0$,

$$s\mathcal{L}\{f(t)\} - f(0) = a\mathcal{L}\{f(t)\} + \frac{1}{s-a} \quad \text{or} \quad (s-a)\mathcal{L}\{f(t)\} = \frac{1}{s-a}$$

i.e.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

THE UNIT STEP FUNCTION

4.17. Prove that $\mathcal{L}\{\mathcal{U}(t-a)\} = \dfrac{e^{-as}}{s}$ if $s > 0$.

We have $\mathcal{U}(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a\end{cases}$ so that

$$\begin{aligned}\mathcal{L}\{\mathcal{U}(t-a)\} &= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt = 0 + \frac{e^{-st}}{-s}\Big|_a^\infty\\ &= \frac{e^{-as}}{s} \quad \text{if } s > 0\end{aligned}$$

4.18. (*a*) Express the function $f(t) = \begin{cases} 8 & t < 2 \\ 6 & t > 2 \end{cases}$ in terms of the unit step function and thus (*b*) obtain its Laplace transform.

(*a*) We have

$$f(t) \;=\; 8 + \begin{cases} 0 & t < 2 \\ -2 & t > 2 \end{cases}$$

$$= \;8 - 2\begin{cases} 0 & t < 2 \\ 1 & t > 2 \end{cases}$$

$$= \;8 - 2\,\mathcal{U}(t-1)$$

(*b*) $\mathcal{L}\{f(t)\} \;=\; \mathcal{L}\{8 - 2u(t-1)\} \;=\; \dfrac{8}{s} - \dfrac{2e^{-s}}{s} \;=\; \dfrac{8 - 2e^{-s}}{s}$

The result can also be obtained directly.

SPECIAL THEOREMS ON LAPLACE TRANSFORMS

4.19. Prove Theorem 4-5, page 101.

We have
$$\mathcal{L}\{f(t)\} \;=\; F(s) \;=\; \int_0^\infty e^{-st} f(t)\,dt$$

Then
$$\mathcal{L}\{e^{at}f(t)\} \;=\; \int_0^\infty e^{-st}[e^{at}f(t)]\,dt \;=\; \int_0^\infty e^{-(s-a)t} f(t)\,dt \;=\; F(s-a)$$

4.20. Prove Theorem 4-6, page 101.

Method 1. Since $\mathcal{U}(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$ we have

$$\begin{aligned}
\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} &= \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,f(t-a)\,dt \\
&= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt \\
&= \int_a^\infty e^{-st} f(t-a)\,dt \;=\; \int_0^\infty e^{-s(v+a)} f(v)\,dv \\
&= e^{-as}\int_0^\infty e^{-sv} f(v)\,dv \\
&= e^{-as}F(s)
\end{aligned}$$

Method 2. Since $F(s) = \int_0^\infty e^{-st} f(t)\,dt$,

$$\begin{aligned}
e^{-as}F(s) &= \int_0^\infty e^{-s(t+a)} f(t)\,dt \;=\; \int_a^\infty e^{-sv} f(v-a)\,dv \\
&= \int_0^a e^{-sv}(0)\,dv + \int_a^\infty e^{-sv} f(v-a)\,dv \\
&= \int_0^\infty e^{-st}\left\{\begin{matrix} 0 & t < a \\ f(t-a) & t > a \end{matrix}\right\} dt \\
&= \int_0^\infty e^{-st} f(t-a)\left\{\begin{matrix} 0 & t < a \\ 1 & t > a \end{matrix}\right\} dt \\
&= \mathcal{L}\{f(t-a)\,\mathcal{U}(t-a)\}
\end{aligned}$$

4.21. Prove Theorem 4-7, page 101.

$$\mathcal{L}\{f(at)\} \;=\; \int_0^\infty e^{-st} f(at)\,dt \;=\; \frac{1}{a}\int_0^\infty e^{-sv/a} f(v)\,dv \;=\; \frac{1}{a}F\left(\frac{s}{a}\right)$$

where we have used the transformation $t = v/a$.

4.22. Prove Theorem 4-8, page 101, for (a) $n = 1$, (b) any positive integer n.

(a) Since $F(s) = \int_0^\infty e^{-st} f(t)\,dt$, we have on differentiating with respect to s and using Leibnitz's rule,

$$\frac{dF}{ds} = F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}[e^{-st} f(t)]\,dt$$

$$= -\int_0^\infty e^{-st}\, tf(t)\,dt = -\mathcal{L}\{tf(t)\}$$

Thus $\mathcal{L}\{tf(t)\} = -F'(s)$.

(b)
$$\frac{d^nF}{ds^n} = \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{\partial^n}{\partial s^n}[e^{-st} f(t)]\,dt$$

$$= (-1)^n \int_0^\infty e^{-st}\,[t^n f(t)]\,dt = (-1)^n \mathcal{L}\{t^n f(t)\}$$

Thus $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$.

4.23. Prove Theorem 4-9, page 101.

We have

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^P e^{-st} f(t)\,dt + \int_P^{2P} e^{-st} f(t)\,dt + \int_{2P}^{3P} e^{-st} f(t)\,dt + \cdots$$

$$= \int_0^P e^{-st} f(t)\,dt + \int_0^P e^{-s(v+P)} f(v+P)\,dv + \int_0^P e^{-s(v+2P)} f(v+2P)\,dv + \cdots$$

Then since $f(t)$ has period $P > 0$, $f(v+P) = f(v)$, $f(v+2P) = f(v)$, etc. Furthermore, we can replace the dummy variable v by t. Thus

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^P e^{-st} f(t)\,dt + e^{-sP}\int_0^P e^{-st} f(t)\,dt + e^{-2sP}\int_0^P e^{-st} f(t)\,dt + \cdots$$

$$= (1 + e^{-sP} + e^{-2sP} + \cdots)\int_0^P e^{-st} f(t)\,dt$$

$$= \frac{1}{1 - e^{-sP}}\int_0^P e^{-st} f(t)\,dt \quad \text{where } s > 0$$

4.24. Prove Theorem 4-10, page 101.

Let $G(t) = \int_0^t f(u)\,du$. Then $G'(t) = f(t)$, $G(0) = 0$. Thus

$$\mathcal{L}\{G'(t)\} = s\mathcal{L}\{G(t)\} - G(0) \quad \text{or} \quad \mathcal{L}\{f(t)\} = s\mathcal{L}\{G(t)\}$$

and so
$$\mathcal{L}\{G(t)\} = \mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

From this we have
$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$$

4.25. Prove Theorem 4-11, page 101.

We shall assume that $\lim_{t\to 0}\frac{f(t)}{t}$ exists [as well as the conditions of Theorem 4-1, page 99], otherwise its Laplace transform may not exist. Then we have if $g(t) = f(t)/t$, or $f(t) = tg(t)$,

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{tg(t)\} = -\frac{d}{ds}\mathcal{L}\{g(t)\} = -\frac{d\,G(s)}{ds} \tag{1}$$

Thus
$$G(s) = -\int_c^s F(u)\,du = \int_s^c F(u)\,du \tag{2}$$

Now since $g(t)$ satisfies the conditions of Theorem 4-1, page 99, it follows that $\lim_{s\to\infty} G(s) = 0$. Then from (2) we see that c must be infinite and so

$$G(s) = \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

4.26. Prove Theorem 4-12, page 102.

We have
$$F(s) = \int_0^\infty e^{-su} f(u)\,du, \qquad G(s) = \int_0^\infty e^{-sv} g(v)\,dv$$

Then
$$\begin{aligned}
F(s)\,G(s) &= \left[\int_0^\infty e^{-su} f(u)\,du\right]\left[\int_0^\infty e^{-sv} g(v)\,dv\right] \\
&= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u)\,g(v)\,du\,dv \\
&= \int_{t=0}^\infty \int_{u=0}^t e^{-st} f(u)\,g(t-u)\,du\,dt \\
&= \int_{t=0}^\infty e^{-st}\left[\int_{u=0}^t f(u)\,g(t-u)\,du\right]dt \\
&= \mathcal{L}\left\{\int_0^t f(u)\,g(t-u)\,du\right\}
\end{aligned}$$

where we have used the transformation $t = u + v$ from the uv plane to the ut plane.

APPLICATIONS OF THE THEOREMS TO FINDING LAPLACE TRANSFORMS

4.27. Find (a) $\mathcal{L}\{e^{3t}\sin 4t\}$, (b) $\mathcal{L}\{t^2 e^{-2t}\}$, (c) $\mathcal{L}\{e^t/\sqrt{t}\}$.

(a) Since $\mathcal{L}\{\sin 4t\} = \dfrac{4}{s^2+16}$, we have by Theorem 4-5,

$$\mathcal{L}\{e^{3t}\sin 4t\} = \frac{4}{(s-3)^2+16} = \frac{4}{s^2-6s+25}$$

(b) Since $\mathcal{L}\{t^2\} = \dfrac{2!}{s^3}$, $\quad \mathcal{L}\{t^2e^{-2t}\} = \dfrac{2!}{(s+2)^3}$

(c) Since $\mathcal{L}\{1/\sqrt{t}\} = \mathcal{L}\{t^{-1/2}\} = \Gamma(1/2)/s^{1/2} = \sqrt{\pi/s}$, $\quad \mathcal{L}\{e^t/\sqrt{t}\} = \sqrt{\pi/(s-1)}$

4.28. Find $\mathcal{L}\{F(t)\}$ where $\quad f(t) = \begin{cases} \sin t & t < \pi \\ t & t > \pi \end{cases}$

We have
$$\begin{aligned}
f(t) &= \sin t + \begin{cases} 0 & t < \pi \\ t - \sin t & t > \pi \end{cases} \\
&= \sin t + (t - \sin t)\mathcal{U}(t-\pi) \\
&= \sin t + [\pi + (t-\pi) + \sin(t-\pi)]\mathcal{U}(t-\pi)
\end{aligned}$$

Then using Theorem 4-6,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin t\} + \mathcal{L}\{[\pi + (t-\pi) + \sin(t-\pi)]\mathcal{U}(t-\pi)\} \\
&= \frac{1}{s^2+1} + e^{-\pi s}\left[\frac{\pi}{s} + \frac{1}{s^2} + \frac{1}{s^2+1}\right]
\end{aligned}$$

This can also be obtained directly without using Theorem 4-6.

4.29. Given that $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$, find $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$.

By Theorem 4-7, page 101,

$$\mathcal{L}\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\left(\frac{1}{s/a}\right) \quad \text{i.e.} \quad \mathcal{L}\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left(\frac{a}{s}\right)$$

4.30. Find (*a*) $\mathcal{L}\{t\sin 2t\}$, (*b*) $\mathcal{L}\{t^2\sin 2t\}$.

By Theorem 4-8, page 101, we have since $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$,

(*a*) $$\mathcal{L}\{t\sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

(*b*) $$\mathcal{L}\{t^2\sin 2t\} = \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right) = \frac{12s^2-16}{(s^2+4)^3}$$

4.31. Find the Laplace transform of the function of period 2π which in the interval $0 \leqq t < 2\pi$ is given by $f(t) = \begin{cases} \sin t & 0 \leqq t < \pi \\ 0 & \pi \leqq t < 2\pi \end{cases}$.

The graph of this function, often called a *rectified sine wave*, is shown in Fig. 4-3.

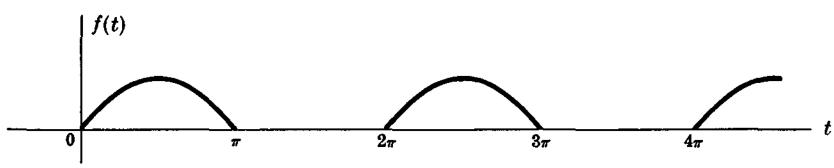

Fig. 4-3

By Theorem 4-9, page 101, the Laplace transform is given by

$$\frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt = \frac{1}{1-e^{-2\pi s}}\left(\frac{1+e^{-\pi s}}{s^2+1}\right) = \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

4.32. Find $\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\}$.

Since $\lim_{t\to 0}\frac{1-e^{-t}}{t} = \lim_{t\to 0}\frac{e^{-t}}{1} = 1$ by L'Hospital's rule and $1-e^{-t}$ is continuous and of exponential order, the conditions of Theorem 4-11 apply. Then since $\mathcal{L}\{1-e^{-t}\} = \frac{1}{s} - \frac{1}{s+1}$, it follows that

$$\begin{aligned}
\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\} &= \int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)du = \lim_{K\to\infty}\int_s^K \left(\frac{1}{u}-\frac{1}{u+1}\right)du \\
&= \lim_{K\to\infty} [\ln u - \ln(u+1)]\Big|_s^K \\
&= \lim_{K\to\infty}\left[\ln\left(1+\frac{1}{s}\right) - \ln\left(1+\frac{1}{K}\right)\right] = \ln\left(1+\frac{1}{s}\right)
\end{aligned}$$

APPLICATIONS OF THE THEOREMS TO FINDING INVERSE LAPLACE TRANSFORMS

4.33. Find (*a*) $\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-2s+5}\right\}$, (*b*) $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+3}}\right\}$, (*c*) $\mathcal{L}^{-1}\left\{\frac{3s-4}{(2s-3)^5}\right\}$.

(*a*) $$\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s-1)+5}{(s-1)^2+4}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}$$

Now since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t, \quad \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \sin 2t$

we have by Theorem 4-5,

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} = e^t \cos 2t, \qquad \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\} = e^t \sin 2t$$

Thus $\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-2s+5}\right\} = 2e^t \cos 2t + \frac{5}{2} e^t \sin 2t = \frac{1}{2}e^t(4\cos 2t + 5\sin 2t)$

(*b*) From Problem 4.12(*d*) we have $\mathcal{L}^{-1}\left\{\frac{1}{s^k}\right\} = \frac{t^{k-1}}{\Gamma(k)}$ or if $k = \frac{1}{2}$, $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{t^{-1/2}}{\sqrt{\pi}}$. Then by Theorem 4-5,

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+3}}\right\} = \frac{e^{-3t}t^{-1/2}}{\sqrt{\pi}}$$

(*c*)

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{3s-4}{(2s-3)^5}\right\} &= \frac{1}{2^5}\mathcal{L}^{-1}\left\{\frac{3s-4}{(s-3/2)^5}\right\} = \frac{1}{32}\mathcal{L}^{-1}\left\{\frac{3(s-3/2)+1/2}{(s-3/2)^5}\right\} \\
&= \frac{3}{32}\mathcal{L}^{-1}\left\{\frac{1}{(s-3/2)^4}\right\} + \frac{1}{64}\mathcal{L}^{-1}\left\{\frac{1}{(s-3/2)^5}\right\} \\
&= \frac{3}{32}\cdot\frac{t^3}{3!}e^{3t/2} + \frac{1}{64}\frac{t^4}{4!}e^{3t/2} = \frac{t^3(t+8)e^{3t/2}}{1536}
\end{aligned}$$

4.34. Find (*a*) $\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+16}\right\}$, (*b*) $\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{\sqrt{s-2}}\right\}$.

(*a*) Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t$, we have by Theorem 4-6,

$$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+16}\right\} = \mathcal{U}(t-2)\cos 4(t-2) = \begin{cases} 0 & t < 2 \\ \cos 4(t-2) & t > 2 \end{cases}$$

(*b*) Since $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{t^{-1/2}}{\sqrt{\pi}}$, $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s-2}}\right\} = \frac{t^{-1/2}e^{2t}}{\sqrt{\pi}}$ and so by Theorem 4-6,

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{\sqrt{s-2}}\right\} = \mathcal{U}(t-5)\,\frac{(t-5)^{-1/2}e^{2(t-5)}}{\sqrt{\pi}} = \begin{cases} 0 & t < 5 \\ \dfrac{(t-5)^{-1/2}e^{2(t-5)}}{\sqrt{\pi}} & t > 5 \end{cases}$$

4.35. Find $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\}$.

Let $F(s) = \ln\left(1+\frac{1}{s}\right)$ so that $F'(s) = \frac{1}{s+1} - \frac{1}{s}$. Then by Theorem 4-8,

$$\mathcal{L}^{-1}\{F'(s)\} = -t\,\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\}$$

or

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s}\right)\right\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s}\right\} = \frac{1-e^{-t}}{t}$$

4.36. Find $\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\}$.

Since $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s+1}}\right\} = \frac{t^{-1/2}e^{-t}}{\sqrt{\pi}}$, we have by Theorem 4-10,

$$\mathcal{L}^{-1}\left\{\frac{1}{s\sqrt{s+1}}\right\} = \int_0^t \frac{u^{-1/2}e^{-u}}{\sqrt{\pi}}\,du = \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-v^2}\,dv$$

on letting $u = v^2$.

4.37. Find $\mathcal{L}^{-1}\left\{\dfrac{1}{(s^2+a^2)^2}\right\}$.

Let $F(s) = G(s) = \dfrac{1}{s^2+a^2}$. Then $f(t) = g(t) = \dfrac{\sin at}{a}$. Thus by the convolution theorem [Theorem 4-12],

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \int_0^t \frac{\sin au}{a}\cdot\frac{\sin a(t-u)}{a}\,du = \frac{1}{a^2}\int_0^t \sin au \sin a(t-u)\,du$$

$$= \frac{1}{2a^2}\int_0^t [\cos a(2u-t) - \cos at]\,du = \frac{1}{2a^3}(\sin at - at\cos at)$$

4.38. Solve for $y(t)$ the equation

$$y(t) = 1 + \int_0^t y(u)\sin(t-u)du$$

Taking the Laplace transform, we have by the convolution theorem,

$$Y(s) = \frac{1}{s} + \mathcal{L}\{y(t)^* \sin t\} = \frac{1}{s} + \frac{Y(s)}{s^2+1}$$

Then $$\left[1 - \frac{1}{s^2+1}\right]Y(s) = \frac{1}{s} \quad \text{or} \quad Y(s) = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}$$

and so $$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{1}{s^3}\right\} = 1 + \frac{t^2}{2}$$

which can be checked as a solution.

The given equation is called an *integral equation* since the unknown function occurs under the integral.

PARTIAL FRACTIONS

4.39. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s-2)(s+1)(s-3)}\right\}$.

$$\frac{2s^2-4}{(s-2)(s+1)(s-3)} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{s-3}$$

To determine constants A, B, C, multiply by $(s-2)(s+1)(s-3)$ so that

$$2s^2 - 4 = A(s+1)(s-3) + B(s-2)(s-3) + C(s-2)(s+1)$$

This must be an identity and thus must hold for all values of s. Then by letting $s = 2, -1, 3$ in succession we find $A = -4/3$, $B = -1/6$, $C = 7/2$. Thus

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s-2)(s+1)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-4/3}{s-2} + \frac{-1/6}{s+1} + \frac{7/2}{s-3}\right\} = -\tfrac{4}{3}e^{2t} - \tfrac{1}{6}e^{-t} + \tfrac{7}{2}e^{3t}$$

4.40. Find $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$.

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \tag{1}$$

or $$3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

Letting $s=1$, we find $A = 2$. Letting $s = 0$, we find $A - C = 1$ so that $C = 1$. Then letting s equal any other number, say -1, we find, $-2 = 2A - 2(C-B)$ and $B = -2$. Thus

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-2s+1}{s^2+1}\right\} = 2e^t - 2\cos t + \sin t$$

Another method.

Multiplying (*1*) by s after finding A and then letting $s \to \infty$, we find $A + B = 0$ or $B = -A = -2$. This method affords some simplification of procedure.

4.41. Find $\mathcal{L}^{-1}\left\{\dfrac{5s^2-15s+7}{(s+1)(s-2)^3}\right\}$.

We have
$$\frac{5s^2-15s+7}{(s+1)(s-2)^3} = \frac{A}{s+1}+\frac{B}{(s-2)^3}+\frac{C}{(s-2)^2}+\frac{D}{s-2} \tag{1}$$
or clearing of fractions
$$5s^2-15s+7 = A(s-2)^3+B(s+1)+C(s+1)(s-2)+D(s+1)(s-2)^2$$

Letting $s=-1$, we find $A=-1$. Letting $s=2$, we find $B=-1$. Letting s equal two other numbers, say 0 and 1, we find $-2C+4D=0$ and $-2C+2D=-2$ from which $C=2$, $D=1$. Thus
$$\mathcal{L}^{-1}\left\{\frac{5s^2-15s+7}{(s+1)(s-2)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+1}+\frac{-1}{(s-2)^3}+\frac{2}{(s-2)^2}+\frac{1}{s-2}\right\}$$
$$= -e^{-t}-\tfrac{1}{2}t^2e^{2t}+2te^{2t}+e^{2t}$$

Another method.

On multiplying (1) by s, after finding A, and letting $s\to\infty$, we find that $A+D=0$ or $D=1$, providing some simplification in the procedure.

SOLUTIONS OF DIFFERENTIAL EQUATIONS

4.42. Solve $y''(t)+y(t)=1$ given $y(0)=1$, $y'(0)=0$.

Take the Laplace transform of both sides of the given differential equation and let $Y=Y(s)=\mathcal{L}\{y(t)\}$. Then
$$\mathcal{L}\{y''(t)+y(t)\} = \mathcal{L}\{1\} \quad\text{or}\quad s^2Y-sy(0)-y'(0)+Y = 1/s$$
Since $y(0)=1$, $y'(0)=0$ this becomes
$$s^2Y-s+Y = \frac{1}{s},\quad (s^2+1)Y = s+\frac{1}{s}\quad\text{or}\quad Y = \frac{s+1/s}{s^2+1} = \frac{1}{s}$$
Thus
$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$
which is the required solution.

4.43. Solve $y''-3y'+2y = 2e^{-t}$, $y(0)=2$, $y'(0)=-1$.

Taking the Laplace transform of the given differential equation,
$$[s^2Y-sy(0)-y'(0)]-3[sY-y(0)]+2Y = \frac{2}{s+1}$$
Then using $y(0)=2$, $y'(0)=-1$ and solving this algebraic equation for Y, we find using partial fractions,
$$Y = \frac{2s^2-5s-5}{(s+1)(s-1)(s-2)} = \frac{1/3}{s+1}+\frac{4}{s-1}+\frac{-7/3}{s-2}$$
Thus taking the inverse Laplace transform, we obtain the required solution
$$y = \tfrac{1}{3}e^{-t}+4e^{t}-\tfrac{7}{3}e^{2t}$$

4.44. Solve $y^{(iv)}+2y''+y = \sin t$, $y(0)=1$, $y'(0)=-2$, $y''(0)=3$, $y'''(0)=0$.

Taking the Laplace transform of the given differential equation and using the initial conditions,
$$[s^4Y-s^3(1)-s^2(-2)-s(3)-0]+2[s^2Y-s(1)-(-2)]+Y = \frac{1}{s^2+1}$$
which can be written
$$(s^4+2s^2+1)Y = \frac{1}{s^2+1}+s^3-2s^2+5s-4$$
or
$$Y = \frac{1}{(s^2+1)^3}+\frac{s^3-2s^2+5s-4}{(s^2+1)^2} = \frac{1}{(s^2+1)^3}+\frac{(s^3+s)-2(s^2+1)+4s-2}{(s^2+1)^2}$$
$$= \frac{1}{(s^2+1)^3}+\frac{s}{s^2+1}-\frac{2}{s^2+1}+\frac{4s-2}{(s^2+1)^2}$$

Now by using the special theorems on Laplace transforms [see Problem 4.78],

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\} = \tfrac{3}{8}\sin t - \tfrac{3}{8}t\cos t - \tfrac{1}{8}t^2\sin t$$

$$\mathcal{L}^{-1}\left\{\frac{4s-2}{(s^2+1)^2}\right\} = 2t\sin t - \sin t + t\cos t$$

and so we find the required solution

$$y = (1+\tfrac{5}{8}t)\cos t - (\tfrac{21}{8} - 2t + \tfrac{1}{8}t^2)\sin t$$

APPLICATIONS TO PHYSICAL PROBLEMS

4.45. Solve Problem 2.32, page 56, by using Laplace transforms.

As in Problem 2.32, the differential equation is

$$\frac{dI}{dt} + 5I = \frac{E}{2}, \qquad I(0) = 0 \tag{1}$$

(*a*) If $E = 40$ the Laplace transform of (*1*) is

$$[s\bar{I} - I(0)] + 5\bar{I} = \frac{20}{s}$$

where $\bar{I} = \mathcal{L}\{I\}$. Then using $I(0) = 0$ and solving for $\bar{I}$, we find

$$\bar{I} = \frac{20}{s(s+5)} = \frac{20}{5}\left(\frac{1}{s} - \frac{1}{s+5}\right) = 4\left(\frac{1}{s} - \frac{1}{s+5}\right)$$

Thus $$I = 4(1 - e^{-5t})$$

(*b*) If $E = 20e^{-3t}$ then

$$[s\bar{I} - I(0)] + 5\bar{I} = \frac{10}{s+3}$$

so that $$\bar{I} = \frac{10}{(s+3)(s+5)} = \frac{10}{2}\left(\frac{1}{s+3} - \frac{1}{s+5}\right) = 5\left(\frac{1}{s+3} - \frac{1}{s+5}\right)$$

Thus $$I = 5(e^{-3t} - e^{-5t})$$

(*c*) If $E = 50\sin 5t$, then

$$[s\bar{I} - I(0)] + 5\bar{I} = \frac{125}{s^2+25}$$

so that $$\bar{I} = \frac{125}{(s+5)(s^2+25)} = \frac{5/2}{s+5} + \frac{(-5/2)s + (25/2)}{s^2+25}$$

Then $$I = \frac{5}{2}e^{-5t} - \frac{5}{2}\cos 5t + \frac{5}{2}\sin 5t$$

4.46. A mass m [Fig. 4-4] is suspended from the end of a vertical spring of constant κ [force required to produce unit stretch]. An external force $F(t)$ acts on the mass as well as a resistive force proportional to the instantaneous velocity. Assuming that x is the displacement of the mass at time t and that the mass starts from rest at $x = 0$, (*a*) set up a differential equation for the motion and (*b*) find x at any time t.

(*a*) The resistive force is given by $-\beta\dfrac{dx}{dt}$. The restoring force is given by $-\kappa x$. Then by Newton's law,

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - \kappa x + F(t)$$

or $$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + \kappa x = F(t) \tag{1}$$

where $$x(0) = 0, \quad x'(0) = 0 \tag{2}$$

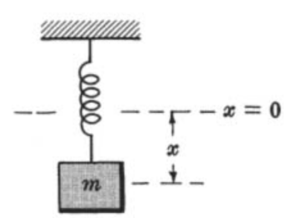

Fig. 4-4

(b) Taking the Laplace transform of (1), using $\mathcal{L}\{F(t)\} = \bar{F}(s)$, $\mathcal{L}\{x\} = X$, we obtain

$$m[s^2X - sx(0) - x'(0)] + \beta[sX - x(0)] + \kappa X = \bar{F}(s)$$

so that on using (2),

$$X = \frac{\bar{F}(s)}{ms^2 + \beta s + \kappa} = \frac{\bar{F}(s)}{m[(s + \beta/2m)^2 + R]} \qquad (3)$$

where $R = \frac{\kappa}{m} - \frac{\beta^2}{4m^2}$. There are 3 cases to be considered.

Case 1, $R > 0$. In this case let $R = \omega^2$. We have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s + \beta/2m)^2 + \omega^2}\right\} = e^{-\beta t/2m}\frac{\sin \omega t}{\omega}$$

Then using the convolution theorem, we find from (3)

$$x = \frac{1}{\omega m}\int_0^t F(u)\, e^{-\beta(t-u)/2m} \sin \omega(t-u)\, du$$

Case 2, $R = 0$. In this case $\mathcal{L}^{-1}\left\{\frac{1}{(s + \beta/2m)^2}\right\} = te^{-\beta t/2m}$ and the convolution theorem in (3) yields

$$x = \frac{1}{m}\int_0^t F(u)(t-u)e^{-\beta(t-u)/m}\, du$$

Case 3, $R < 0$. In this case let $R = -\alpha^2$. We have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s + \beta/2m)^2 - \alpha^2}\right\} = e^{-\beta t/2m}\frac{\sinh \alpha t}{\alpha}$$

Then using the convolution theorem (3) yields

$$x = \frac{1}{\alpha m}\int_0^t F(u)e^{-\beta(t-u)/2m} \sinh \alpha(t-u)\, du$$

4.47. Solve Problem 3.48, page 93, by Laplace transforms.

From equations (1) and (2) of Problem 3.48 we have on taking Laplace transforms, using $\mathcal{L}\{I_1\} = \bar{I}_1$, $\mathcal{L}\{I_2\} = \bar{I}_2$, $I = I_1 + I_2$, $\mathcal{L}\{I\} = \bar{I}_1 + \bar{I}_2$,

$$20(\bar{I}_1 + \bar{I}_2) - \frac{120}{s} + 2[s\bar{I}_1 - \bar{I}_1(0)] + 10\bar{I}_1 = 0$$

$$-10\bar{I}_1 - 2[s\bar{I}_1 - I_1(0)] + 4[s\bar{I}_2 - I_2(0)] + 20\bar{I}_2 = 0$$

Using $I_1(0) = 0$, $I_2(0) = 0$, these become

$$(30 + 2s)\bar{I}_1 + 20\bar{I}_2 = 120/s$$

$$(-10 - 2s)\bar{I}_1 + (4s + 20)\bar{I}_2 = 0$$

Solving,

$$\bar{I}_1 = \frac{\begin{vmatrix} 120/s & 20 \\ 0 & 4s + 20 \end{vmatrix}}{\begin{vmatrix} 30 + 2s & 20 \\ -10 - 2s & 4s + 20 \end{vmatrix}}, \qquad \bar{I}_2 = \frac{\begin{vmatrix} 30 + 2s & 120/s \\ -10 - 2s & 0 \end{vmatrix}}{\begin{vmatrix} 30 + 2s & 20 \\ -10 - 2s & 4s + 20 \end{vmatrix}}$$

or $$\bar{I}_1 = \frac{60}{s(s + 20)} = 3\left(\frac{1}{s} - \frac{1}{s + 20}\right), \qquad \bar{I}_2 = \frac{30}{s(s + 20)} = \frac{3}{2}\left(\frac{1}{s} - \frac{1}{s + 20}\right)$$

Then $I_1 = 3(1 - e^{-20t})$, $I_2 = \frac{3}{2}(1 - e^{-20t})$, $I = I_1 + I_2 = \frac{9}{2}(1 - e^{-20t})$.

Supplementary Problems

LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

4.48. Find the Laplace transforms of each of the following:

(a) $4e^{2t/3}$ (e) $(e^{3t}-e^{-3t})^2$ (h) $\sin 2t \cos 2t$

(b) $6t-3$ (f) $(\sqrt{t}+1)(2-\sqrt{t})/\sqrt{t}$ (i) $(\sqrt[3]{t^2}-1/\sqrt[3]{t})^2$

(c) $(t+1)^2$ (g) $8\sin^2 3t$ (j) $5\sinh 2t - 5\cosh 2t$

(d) $2\sin 3t + 5\cos 3t$

4.49. Find $\mathcal{L}\{f(t)\}$ in each case: (a) $f(t) = \begin{cases} -1 & 0 \leqq t \leqq 4 \\ 1 & t > 4 \end{cases}$ (b) $f(t) = \begin{cases} t+1 & 0 \leqq t < 3 \\ 0 & t \geqq 3 \end{cases}$

(c) $f(t) = \begin{cases} 0 & 0 \leqq t < 2 \\ 1 & 2 \leqq t < 4 \\ 0 & t \geqq 4 \end{cases}$

4.50. Prove that e^{t^2} does not have a Laplace transform.

4.51. Determine whether $\sin t^2$ has a Laplace transform and justify your conclusions.

4.52. Find (a) $\mathcal{L}\{10 \sin 3t \cos 5t\}$, (b) $\mathcal{L}\{f(t)\}$ if $f(t) = \begin{cases} \sin t & 0 \leqq t < \pi \\ 0 & t \geqq \pi \end{cases}$.

ELEMENTARY INVERSE LAPLACE TRANSFORMS

4.53. Find (a) $\mathcal{L}^{-1}\left\{\frac{2}{s-3}\right\}$ (d) $\mathcal{L}^{-1}\left\{\frac{1}{4s^2+9}\right\}$ (g) $\mathcal{L}^{-1}\left\{\frac{2-s}{s^{3/2}}\right\}$ (i) $\mathcal{L}^{-1}\left\{\frac{(2s+1)^2}{s^5}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{1}{3s+5}\right\}$ (e) $\mathcal{L}^{-1}\left\{\frac{2s-8}{s^2+36}\right\}$ (h) $\mathcal{L}^{-1}\left\{\frac{3s-16}{s^2-64}\right\}$ (j) $\mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s+5)}\right\}$

(c) $\mathcal{L}^{-1}\left\{\frac{4s}{s^2+16}\right\}$ (f) $\mathcal{L}^{-1}\left\{\frac{s^3-s^2+s-1}{s^5}\right\}$

4.54. Find $\mathcal{L}^{-1}\left\{\frac{s^2+2}{(s^2+10)(s^2+20)}\right\}$.

LAPLACE TRANSFORMS OF DERIVATIVES

4.55. Verify the result $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(s)\} - f(0)$ for each of the following: (a) $f(t) = 3e^{2t}$, (b) $f(t) = \cos 5t$, (c) $f(t) = t^2+2t-4$.

4.56. Verify the result $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$ for each of the functions in Problem 4.55.

4.57. If $f(t) = t \sin at$, (a) show that $f''(t) + a^2 f(t) = 2a \cos at$ and thus (b) find $\mathcal{L}\{f(t)\}$.

4.58. Does the result $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ hold for

(a) $f(t) = \sqrt{t}$, (b) $f(t) = 1/\sqrt{t}$, (c) $f(t) = \begin{cases} t & t > 0 \\ 5 & t = 0 \end{cases}$? Explain.

4.59. Prove that $\mathcal{L}\{f'''(t)\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$, giving conditions under which it holds.

THE UNIT STEP FUNCTION

4.60. Find (a) $\mathcal{L}\{2\mathcal{U}(t-1) + 3\mathcal{U}(t-2)\}$, (b) $\mathcal{L}\{t\mathcal{U}(t-3)\}$ and graph each of the given functions of t.

4.61. Discuss the significance of (a) $[\mathcal{U}(t-a)]^2$, (b) $[\mathcal{U}(t-a)]^p$ where p is any positive integer.

4.62. Find (a) $\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{2e^{-s}-e^{-2s}}{s}\right\}$ and graph.

4.63. (a) Express the function $f(t) = \begin{cases} \cos t & t > \pi/2 \\ 3 & t < \pi/2 \end{cases}$ in terms of the unit step function and (b) find its Laplace transform.

MISCELLANEOUS LAPLACE TRANSFORMS

4.64. Find (a) $\mathcal{L}\{t^3e^{-5t}\}$, (b) $\mathcal{L}\{e^{-t}\cos 2t\}$, (c) $\mathcal{L}\{\sqrt{t}\,e^{4t}\}$, (d) $\mathcal{L}\{e^{-3t}/\sqrt{t}\}$.

4.65. Find (a) $\mathcal{L}\{t \sin 3t\}$, (b) $\mathcal{L}\{t^2 \cos 2t\}$.

4.66. Find $\mathcal{L}\{te^t \sin t\}$.

4.67. (a) Graph the function $f(t) = t,\ 0 \leqq t < 4$ which is extended periodically with period 4 and (b) find the Laplace transform of this function.

4.68. Find $\mathcal{L}\{f(t)\}$ if $f(t) = e^{-t},\ 0 \leqq t < 2$ and $f(t+2) = f(t)$.

4.69. Verify that $\mathcal{L}\left\{\int_0^t \cos 2u\, du\right\} = \frac{1}{s}\mathcal{L}\{\cos 2t\}$.

4.70. Show that $\mathcal{L}\left\{\int_0^t \int_0^{t_1} f(u)\, du\right\} = \frac{F(s)}{s^2}$ where $F(s) = \mathcal{L}\{f(t)\}$ and generalize.

4.71. Find (a) $\mathcal{L}\left\{\frac{e^{-3t} - e^{-5t}}{t}\right\}$, (b) $\mathcal{L}\left\{\frac{\cos 2t - \cos 3t}{t}\right\}$, (c) $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$.

4.72. Find $\mathcal{L}\left\{\frac{1-\cos t}{t^2}\right\}$.

4.73. Show that (a) $\int_0^\infty \frac{e^{-2t} - e^{-4t}}{t}\, dt = \ln 2$, (b) $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

4.74. Evaluate (a) $t*e^t$ and (b) $\mathcal{L}\{t*e^t\}$.

4.75. Prove that (a) $f*g = g*f$, (b) $f*(g+h) = f*g + f*h$, (c) $f*(g*h) = (f*g)*h$ and discuss the significance of the results.

MISCELLANEOUS INVERSE LAPLACE TRANSFORMS

4.76. Find (a) $\mathcal{L}^{-1}\left\{\frac{3s+9}{s^2+2s+10}\right\}$ (c) $\mathcal{L}^{-1}\left\{\frac{1}{(s-5)^3}\right\}$ (e) $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+\pi^2}\right\}$.

(b) $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{4s-1}}\right\}$ (d) $\mathcal{L}^{-1}\left\{\frac{s}{(s-5)^3}\right\}$

4.77. Find (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\}$ (c) $\mathcal{L}^{-1}\left\{\frac{8-10s}{(s+1)(s-2)^2}\right\}$ (e) $\mathcal{L}^{-1}\left\{\frac{150}{(s^2+2s+5)(s^2-4s+8)}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}$ (d) $\mathcal{L}^{-1}\left\{\frac{3s-12}{(s^2+8)(s-1)}\right\}$ (f) $\mathcal{L}^{-1}\left\{\frac{1}{s^4+4}\right\}$.

4.78. Find (a) $\mathcal{L}^{-1}\left\{\frac{1}{s^4-2s^3}\right\}$, (b) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+1)^2}\right\}$, (c) $\mathcal{L}^{-1}\left\{\frac{4s-2}{(s^2+1)^2}\right\}$, (d) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\}$.

4.79. Find $\mathcal{L}^{-1}\left\{\ln\left(\frac{s+6}{s+2}\right)\right\}$.

4.80. Use the series for e^u to show that $\mathcal{L}^{-1}\left\{\frac{e^{-1/s}}{\sqrt{s}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$.

4.81. Solve the integral equations (a) $y(t) + \int_0^t y(t-u)e^u\, du = 2t - 3$, (b) $\int_0^t y(u)\, y(t-u)\, du = t^2e^{-t}$.

4.82. Use Laplace transforms to evaluate $\int_0^\infty \sin x^2\, dx$. [*Hint.* Consider $\int_0^\infty \sin tx^2\, dx$.]

SOLUTIONS OF DIFFERENTIAL EQUATIONS

4.83. Solve each of the following:

(a) $y''(t) - 3y'(t) + 2y(t) = 4; \quad y(0) = 1, \ y'(0) = 0$

(b) $y''(t) + 16y(t) = 32t; \quad y(0) = 3, \ y'(0) = -2$

(c) $y''(t) + 4y'(t) + 4y(t) = 6e^{-2t}; \quad y(0) = -2, \ y'(0) = 8$

(d) $y'''(t) + y'(t) = t + 1$

4.84. Solve $y'''(t) + 8y(t) = 32t^3 - 16t$ if $y(0) = y'(0) = y''(0) = 0$.

4.85. Solve the simultaneous equations

$$\begin{cases} 2x(t) - y(t) - y'(t) = 4(1 - e^{-t}) \\ 2x'(t) + y(t) = 2(1 + 3e^{-2t}) \end{cases}$$

subject to the conditions $x(0) = y(0) = 0$.

4.86. Solve (a) Problem 3.76, (b) Problem 3.77, (c) Problem 3.78, (d) Problem 3.79 on page 95 by Laplace transforms.

APPLICATIONS TO PHYSICAL PROBLEMS

4.87. Solve (a) Problem 3.80, (b) Problem 3.81, (c) Problem 3.83, (d) Problem 3.84, page 95, by Laplace transforms.

4.88. Use Laplace transforms to find the charge and current at any time in a series circuit having an inductance L, capacitance C, resistance R and e.m.f. $E(t)$. Treat all cases assuming that the initial charge and current are zero.

4.89. Solve Problem 3.85, page 95, by Laplace transforms.

4.90. (a) A particle of mass m moves along the x axis in such a way that the force acting is given by $F(t) = \begin{cases} F_0/\epsilon & 0 < t < \epsilon \\ 0 & t > \epsilon \end{cases}$. Assuming that it starts from rest at the origin, find the position of the particle at any time and interpret physically. (b) Discuss the result in (a) if the limit is taken as $\epsilon \to 0$.

4.91. Let $f(t)$ be any continuous function. Suppose there exists a function $\delta(t)$ such that

$$\int_0^\infty f(t)\,\delta(t - t_0)\,dt = f(t_0)$$

Show that (a) $\mathcal{L}\{\delta(t)\} = 1$, (b) $\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$, (c) $\mathcal{L}\{\delta'(t)\} = s$. We often call $\delta(t)$ the *Dirac delta function*).

4.92. In Problem 4.90 replace the force by $F(t) = F_0\,\delta(t)$ and solve. Discuss a possible relationship between $\delta(t)$ and the function $\begin{cases} 1/\epsilon & 0 < t < \epsilon \\ 0 & t > \epsilon \end{cases}$. [*Hint.* Examine the case where $\epsilon \to 0$.]

4.93. Graph the function $F(t) = \begin{cases} 1/\epsilon^2 & 0 < t < \epsilon \\ -1/\epsilon^2 & \epsilon < t < 2\epsilon \end{cases}$. Find (a) $\mathcal{L}\{F(t)\}$ and (b) $\lim_{\epsilon \to 0} \mathcal{L}\{F(t)\}$. Discuss the relationship of this problem with $\mathcal{L}^{-1}\{s^2\}$.

Answers to Supplementary Problems

4.48. (a) $\dfrac{12}{3s - 2}$ (b) $\dfrac{6 - 3s}{s^2}$ (c) $\dfrac{s^2 + 2s + 2}{s^3}$ (d) $\dfrac{5s + 6}{s^2 + 9}$ (e) $\dfrac{72}{s(s^2 - 36)}$ (f) $\dfrac{1}{s} + \dfrac{2\sqrt{\pi}}{\sqrt{s}} - \dfrac{3\sqrt{\pi}}{4s^{5/2}}$ (g) $\dfrac{144}{s(s^2 + 36)}$ (h) $\dfrac{2}{s^2 + 16}$ (i) $\dfrac{4\Gamma(1/3)}{9s^{7/3}} - \dfrac{2\Gamma(1/3)}{3s^{4/3}} + \dfrac{\Gamma(2/3)}{s^{2/3}}$ (j) $\dfrac{-5}{s + 2}$

4.49. (a) $\dfrac{2e^{-4s} - 1}{s}$ (b) $\dfrac{1}{s} + \dfrac{1}{s^2} - \dfrac{4e^{-3s}}{s} - \dfrac{e^{-3s}}{s^2}$ (c) $\dfrac{e^{-2s} - e^{-4s}}{s}$

4.52. (*a*) $\dfrac{40}{s^2+64} - \dfrac{10}{s^2+4}$ (*b*) $\dfrac{1+e^{-\pi s}}{s^2+1}$

4.53. (*a*) $2e^{3t}$ (*d*) $\frac{1}{6}\sin\frac{3}{2}t$ (*g*) $\dfrac{4t^{1/2}-t^{-1/2}}{\sqrt{\pi}}$ (*i*) $2t^2+\frac{3}{8}t^3+\dfrac{t^4}{24}$

(*b*) $\frac{1}{3}e^{-5t/3}$ (*e*) $2\cos 6t - \frac{4}{3}\sin 6t$ (*h*) $3\cosh 8t - 2\sinh 8t$ (*j*) $\frac{5}{2}e^{-5t} - \frac{3}{2}e^{-3t}$

(*c*) $4\cos 4t$ (*f*) $t - \dfrac{t^2}{2} + \dfrac{t^3}{6} - \dfrac{t^4}{24}$

4.54. $\dfrac{9}{10\sqrt{5}}\sin 2\sqrt{5}\,t - \dfrac{4}{5\sqrt{10}}\sin\sqrt{10}\,t$

4.60. (*a*) $\dfrac{2e^{-s}+3e^{-2s}}{s}$ (*b*) $\dfrac{e^{-3s}(3s+1)}{s^2}$

4.62. (*a*) $\mathcal{U}(t-5)$ (*b*) $2\mathcal{U}(t-1) - \mathcal{U}(t-2)$

4.63. (*a*) $3 + (\cos t - 3)\mathcal{U}(t-\pi/2)$ (*b*) $\dfrac{3(1-e^{-\pi s/2})}{s} - \dfrac{e^{-\pi s/2}}{s^2+1}$

4.64. (*a*) $\dfrac{6}{(s+5)^4}$ (*b*) $\dfrac{s+1}{s^2+2s+5}$ (*c*) $\dfrac{\sqrt{\pi}}{2(s-4)^{3/2}}$ (*d*) $\sqrt{\dfrac{\pi}{s+3}}$

4.65. (*a*) $\dfrac{6s}{(s^2+9)^2}$ (*b*) $\dfrac{2s^3-24s}{(s^2+4)^3}$ **4.66.** $\dfrac{2s-2}{(s^2-2s+2)^2}$

4.67. (*b*) $\dfrac{1-4se^{-4s}-e^{-4s}}{s^2(1-e^{-4s})}$ **4.68.** $\dfrac{1-e^{-2(s+1)}}{(s+1)(1-e^{-2s})}$

4.71. (*a*) $\ln\left(\dfrac{s+5}{s+3}\right)$ (*b*) $\dfrac{1}{2}\ln\left(\dfrac{s^2+9}{s^2+4}\right)$ (*c*) $\tan^{-1}(1/s)$

4.72. $s\ln(s/\sqrt{s^2+1}) + \dfrac{\pi}{2} - \tan^{-1}s$

4.74. (*a*) $e^t - 1 - t$ (*b*) $\dfrac{1}{s^2(s-1)}$

4.76. (*a*) $e^{-t}(3\cos 3t + 2\sin 3t)$ (*c*) $\frac{1}{2}t^2e^{5t}$ (*d*) $te^{5t} + \frac{5}{2}t^2e^{5t}$

(*b*) $t^{-1/2}e^{t/4}/2\sqrt{\pi}$

(*e*) $\mathcal{U}(t-3)\sin\pi(t-3)$ or $-\mathcal{U}(t-3)\sin\pi t = \begin{cases} 0 & t<3 \\ -\sin\pi t & t>3 \end{cases}$

4.77. (*a*) $\frac{1}{2}\sinh 2t$ (*b*) $\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$ (*c*) $2e^{-t} - 2e^{2t} - 4te^{2t}$

(*d*) $\cos 2\sqrt{2}\,t + \sqrt{2}\sin 2\sqrt{2}\,t - e^t$

(*e*) $e^{-t}(4\cos 2t + 3\sin 2t) + e^{2t}(3\sin 2t - 4\cos 2t)$

(*f*) $\frac{1}{4}(\sin t\cosh t - \cos t\sinh t)$

4.78. (*a*) $\frac{1}{8}(e^{2t} - 2t^2 - 2t - 1)$ (*c*) $2t\sin t - \sin t + t\cos t$

(*b*) $\frac{1}{2}(\sin t + t\cos t)$ (*d*) $\frac{1}{8}(3\sin t - 3t\cos t - t^2\sin t)$

4.79. $\dfrac{e^{-2t}-e^{-6t}}{t}$ **4.81.** (*a*) $y(t) = 5t - 3 - t^2$ (*b*) $y(t) = 2\sqrt{2t/\pi}\,e^{-t}$ **4.82.** $\frac{1}{2}\sqrt{\pi}/2$

4.83. (*a*) $y(t) = 2 - 2e^t + e^{2t}$ (*c*) $y(t) = (3t^2+4t-2)e^{-2t}$

(*b*) $y(t) = 2\sin 4t + 3\cos 4t + 2t$ (*d*) $y(t) = c_1 + c_2\cos t + c_3\sin t + \frac{1}{2}t^2 + t$

4.84. $y(t) = 4t^3 - 2t - 3 + \frac{8}{3}e^{-2t} + \frac{1}{3}e^t(7\cos\sqrt{3}\,t + \sqrt{3}\sin\sqrt{3}\,t)$

4.85. $x(t) = 3 - 2e^{-t} - e^{-2t}$, $y(t) = 2 - 4e^{-t} + 2e^{-2t}$

4.90. (*a*) $x = \dfrac{F_0t^2}{2m\epsilon}\{1 - \mathcal{U}(t-\epsilon)\}$