Chapter 7

Fourier Series

PERIODIC FUNCTIONS

A function $f(x)$ is said to have a *period* T or to be *periodic* with period T if for all x, $f(x+T) = f(x)$, where T is a positive constant. The least value of $T > 0$ is called the *least period* or simply *the period* of $f(x)$.

Example 1. The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots$, since $\sin(x+2\pi), \sin(x+4\pi), \sin(x+6\pi), \ldots$ all equal $\sin x$. However, 2π is the *least period* or *the period* of $\sin x$.

Example 2. The period of $\sin nx$ or $\cos nx$, where n is a positive integer, is $2\pi/n$.

Example 3. The period of $\tan x$ is π.

Example 4. A constant has any positive number as period.

Other examples of periodic functions are shown in the graphs of Figures 7-1(a), (b) and (c) below.

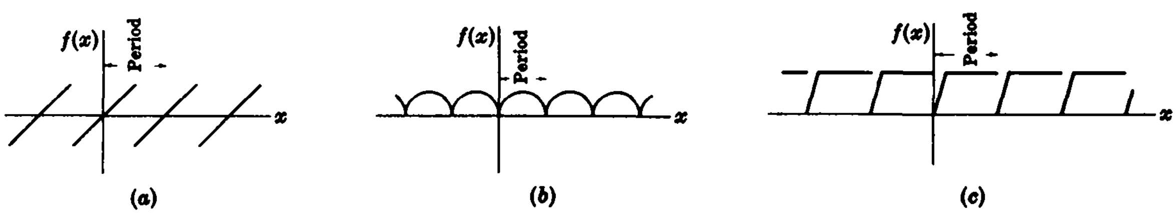

Fig. 7-1

FOURIER SERIES

Let $f(x)$ be defined in the interval $(-L, L)$ and outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where the *Fourier coefficients* a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2}$$

If $f(x)$ has the period $2L$, the coefficients a_n and b_n can be determined equivalently from

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{c}^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{c}^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx \end{cases} \tag{3}$$

where c is any real number. In the special case $c = -L$, (*3*) becomes (*2*).

To determine a_0 in (*1*), we use (*2*) or (*3*) with $n = 0$. For example, from (*2*) we see that $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$. Note that the constant term in (*1*) is equal to $\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$, which is the *mean* of $f(x)$ over a period.

If $L = \pi$, the series (*1*) and the coefficients (*2*) or (*3*) are particularly simple. The function in this case has the period 2π.

DIRICHLET CONDITIONS

Theorem 7-1. Suppose that

(1) $f(x)$ is defined and single-valued except possibly at a finite number of points in $(-L, L)$

(2) $f(x)$ is periodic outside $(-L, L)$ with period $2L$

(3) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

Then the series (*1*) with coefficients (*2*) or (*3*) converges to

(*a*) $f(x)$ if x is a point of continuity

(*b*) $\dfrac{f(x+0) + f(x-0)}{2}$ if x is a point of discontinuity

In this theorem $f(x+0)$ and $f(x-0)$ are the *right* and *left hand limits* of $f(x)$ at x and represent $\lim_{\epsilon \to 0} f(x+\epsilon)$ and $\lim_{\epsilon \to 0} f(x-\epsilon)$ respectively where $\epsilon > 0$. These are often written $\lim_{\epsilon \to 0+} f(x+\epsilon)$ and $\lim_{\epsilon \to 0+} f(x-\epsilon)$ to emphasize that ϵ is approaching zero through positive values. For a proof see Problems 7.18-7.23.

The conditions (1), (2) and (3) imposed on $f(x)$ are *sufficient* but not necessary, and are generally satisfied in practice. There are at present no known necessary and sufficient conditions for convergence of Fourier series. It is of interest that continuity of $f(x)$ does not *alone* insure convergence of a Fourier series.

ODD AND EVEN FUNCTIONS

A function $f(x)$ is called *odd* if $f(-x) = -f(x)$. Thus x^3, $x^5 - 3x^3 + 2x$, $\sin x$, $\tan 3x$ are odd functions.

A function $f(x)$ is called *even* if $f(-x) = f(x)$. Thus x^4, $2x^6 - 4x^2 + 5$, $\cos x$, $e^x + e^{-x}$ are even functions.

The functions portrayed graphically in Figures 7-1(*a*) and 7-1(*b*) are odd and even respectively, but that of Fig. 7-1(*c*) is neither odd nor even.

In the Fourier series corresponding to an odd function, only sine terms can be present. In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant which we shall consider a cosine term) can be present.

HALF RANGE FOURIER SINE OR COSINE SERIES

A half range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present respectively. When a half range series corresponding to a given function is desired, the function is generally defined in the interval $(0, L)$ [which is half of the interval $(-L, L)$, thus accounting for the name *half range*] and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely $(-L, 0)$. In such case, we have

$$\begin{cases} a_n = 0, \quad b_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx & \text{for } \textit{half range sine series} \\ b_n = 0, \quad a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx & \text{for } \textit{half range cosine series} \end{cases} \tag{4}$$

PARSEVAL'S IDENTITY states that

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx \;=\; \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{5}$$

if a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$ satisfies the Dirichlet conditions.

DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

Differentiation and integration of Fourier series can be justified by using the theorems on page 7 which hold for series in general. It must be emphasized, however, that those theorems provide sufficient conditions and are not necessary. The following theorem for integration is especially useful.

Theorem 7-2. The Fourier series corresponding to $f(x)$ may be integrated term by term from a to x, and the resulting series will converge uniformly to $\int_a^x f(u)\,du$ provided that $f(x)$ is piecewise continuous in $-L \leqq x \leqq L$ and both a and x are in this interval.

COMPLEX NOTATION FOR FOURIER SERIES

Using Euler's identities,

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad e^{-i\theta} = \cos\theta - i\sin\theta \tag{6}$$

where $i = \sqrt{-1}$ [see Problem 1.61, page 30], the Fourier series for $f(x)$ can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \tag{7}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\,e^{-in\pi x/L}\,dx \tag{8}$$

In writing the equality *(7)*, we are supposing that the Dirichlet conditions are satisfied and further that $f(x)$ is continuous at x. If $f(x)$ is discontinuous at x, the left side of *(7)* should be replaced by $\dfrac{f(x+0)+f(x-0)}{2}$.

ORTHOGONAL FUNCTIONS

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are called *orthogonal* (perpendicular) if $\mathbf{A}\cdot\mathbf{B} = 0$ or $A_1B_1 + A_2B_2 + A_3B_3 = 0$, where $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$. Although not geometrically or physically evident, these ideas can be generalized to include vectors with more than three components. In particular we can think of a function, say $A(x)$, as being a vector with an *infinity of components* (i.e. an *infinite dimensional vector*), the value of each component being specified by substituting a particular value of x in some interval (a, b). It is natural in such case to define two functions, $A(x)$ and $B(x)$, as *orthogonal* in (a, b) if

$$\int_a^b A(x)\,B(x)\,dx \quad = \quad 0 \tag{9}$$

A vector $\mathbf{A}$ is called a *unit vector* or *normalized vector* if its magnitude is unity, i.e. if $\mathbf{A}\cdot\mathbf{A} = A^2 = 1$. Extending the concept, we say that the function $A(x)$ is *normal* or *normalized* in (a, b) if

$$\int_a^b \{A(x)\}^2\,dx \quad = \quad 1 \tag{10}$$

From the above it is clear that we can consider a set of functions $\{\phi_k(x)\}$, $k = 1, 2, 3, \ldots$, having the properties

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx \quad = \quad 0 \qquad m \neq n \tag{11}$$

$$\int_a^b \{\phi_m(x)\}^2\,dx \quad = \quad 1 \qquad m = 1, 2, 3, \ldots \tag{12}$$

In such case, each member of the set is orthogonal to every other member of the set and is also normalized. We call such a set of functions an *orthonormal set* in (a, b).

The equations *(11)* and *(12)* can be summarized by writing

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx \quad = \quad \delta_{mn} \tag{13}$$

where δ_{mn}, called *Kronecker's symbol*, is defined as 0 if $m \neq n$ and 1 if $m = n$.

Just as any vector $\mathbf{r}$ in 3 dimensions can be expanded in a set of mutually orthogonal unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the form $\mathbf{r} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, so we consider the possibility of expanding a function $f(x)$ in a set of orthonormal functions, i.e.,

$$f(x) \quad = \quad \sum_{n=1}^{\infty} c_n \phi_n(x) \qquad a \leqq x \leqq b \tag{14}$$

Such series, called *orthonormal series*, are generalizations of Fourier series and are of great interest and utility both from theoretical and applied viewpoints.

If

$$\int_a^b w(x)\,\psi_m(x)\,\psi_n(x)\,dx \quad = \quad \delta_{mn} \tag{15}$$

where $w(x) \geqq 0$, we often say that $\psi_m(x)$ and $\psi_n(x)$ are orthonormal with respect to the *density function* or *weight function* $w(x)$. In such case the set of functions $\{\sqrt{w(x)}\,\phi_n(x)\}$ is an orthonormal set in (a, b).

Solved Problems

FOURIER SERIES

7.1. Graph each of the following functions.

(a) $f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases}$ Period = 10

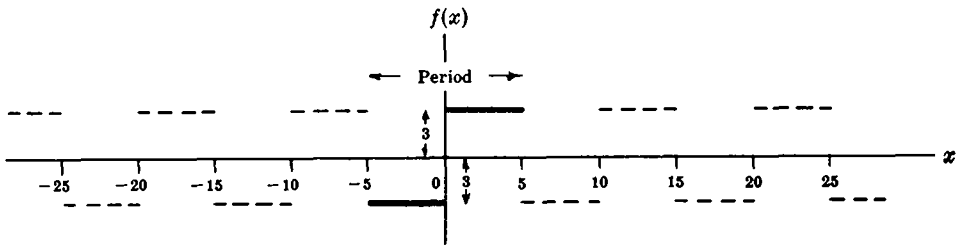

Fig. 7-2

Since the period is 10, that portion of the graph in $-5 < x < 5$ (indicated heavy in Fig. 7-2 above) is extended periodically outside this range (indicated dashed). Note that $f(x)$ is not defined at $x = 0, 5, -5, 10, -10, 15, -15$, etc. These values are the *discontinuities* of $f(x)$.

(b) $f(x) = \begin{cases} \sin x & 0 \leqq x \leqq \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period = 2π

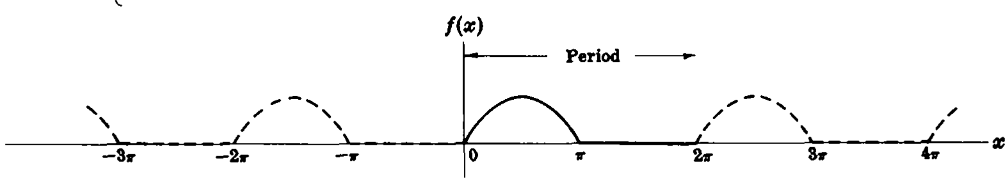

Fig. 7-3

Refer to Fig. 7-3 above. Note that $f(x)$ is defined for all x and is continuous everywhere.

(c) $f(x) = \begin{cases} 0 & 0 \leqq x < 2 \\ 1 & 2 \leqq x < 4 \\ 0 & 4 \leqq x < 6 \end{cases}$ Period = 6

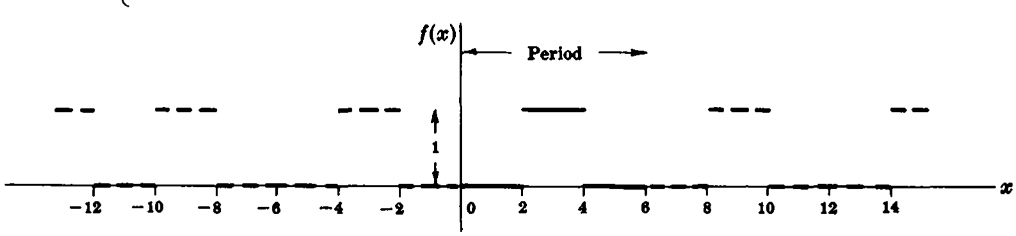

Fig. 7-4

Refer to Fig. 7-4 above. Note that $f(x)$ is defined for all x and is discontinuous at $x = \pm 2, \pm 4, \pm 8, \pm 10, \pm 14, \ldots$.

7.2. Prove $\displaystyle\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = \int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = 0$ if $k = 1, 2, 3, \ldots$.

$$\int_{-L}^{L} \sin\frac{k\pi x}{L}\,dx = -\frac{L}{k\pi}\cos\frac{k\pi x}{L}\bigg|_{-L}^{L} = -\frac{L}{k\pi}\cos k\pi + \frac{L}{k\pi}\cos(-k\pi) = 0$$

$$\int_{-L}^{L} \cos\frac{k\pi x}{L}\,dx = \frac{L}{k\pi}\sin\frac{k\pi x}{L}\bigg|_{-L}^{L} = \frac{L}{k\pi}\sin k\pi - \frac{L}{k\pi}\sin(-k\pi) = 0$$

7.3. Prove (a) $\displaystyle\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & m \neq n \\ L & m = n\end{cases}$

(b) $\displaystyle\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$

where m and n can assume any of the values $1, 2, 3, \ldots$.

(a) From trigonometry: $\cos A \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$, $\sin A \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$.

Then, if $m \neq n$, we have by Problem 7.2,

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\cos\frac{(m-n)\pi x}{L} + \cos\frac{(m+n)\pi x}{L}\right\}dx = 0$$

Similarly if $m \neq n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L}\right\}dx = 0$$

If $m = n$, we have

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 + \cos\frac{2n\pi x}{L}\right)dx = L$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\frac{2n\pi x}{L}\right)dx = L$$

Note that if $m = n = 0$ these integrals are equal to $2L$ and 0 respectively.

(b) We have $\sin A \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$. Then by Problem 7.2, if $m \neq n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left\{\sin\frac{(m-n)\pi x}{L} + \sin\frac{(m+n)\pi x}{L}\right\}dx = 0$$

If $m = n$,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L} \sin\frac{2n\pi x}{L}\,dx = 0$$

The results of parts (a) and (b) remain valid even when the limits of integration $-L, L$ are replaced by $c, c+2L$ respectively.

7.4. If the series $\displaystyle A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots$,

(a) $\displaystyle a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, (b) $\displaystyle b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, (c) $A = \dfrac{a_0}{2}$.

(a) Multiplying

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \tag{1}$$

by $\cos\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 7.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \quad \text{if } m \neq 0\end{aligned} \tag{2}$$

Thus

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(b) Multiplying (1) by $\sin\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 7.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= b_m L\end{aligned}$$

Thus
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1,2,3,\ldots$$

(c) Integration of (1) from $-L$ to L, using Problem 7.2, gives

$$\int_{-L}^{L} f(x)\,dx = 2AL \qquad \text{or} \qquad A = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Putting $m = 0$ in the result of part (a), we find $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ and so $A = \frac{a_0}{2}$.

The above results also hold when the integration limits $-L, L$ are replaced by $c, c+2L$.

Note that in all parts above, interchange of summation and integration is valid because the series is *assumed* to converge uniformly to $f(x)$ in $(-L, L)$. Even when this assumption is not warranted, the coefficients a_m and b_m as obtained above are called *Fourier coefficients* corresponding to $f(x)$, and the corresponding series with these values of a_m and b_m is called the *Fourier series* corresponding to $f(x)$. An important problem in this case is to investigate conditions under which this series actually converges to $f(x)$. Sufficient conditions for this convergence are the *Dirichlet conditions* established below.

7.5. (a) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

(b) Write the corresponding Fourier series.

(c) How should $f(x)$ be defined at $x = -5$, $x = 0$ and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leqq x \leqq 5$?

The graph of $f(x)$ is shown in Fig. 7-5 below.

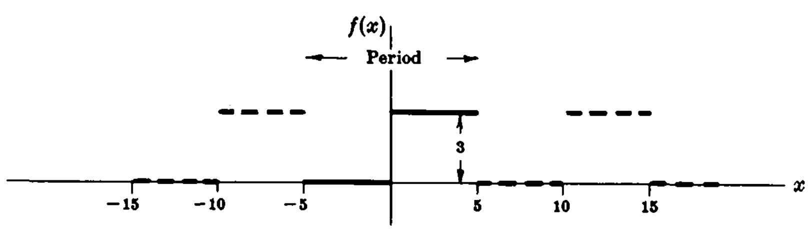

Fig. 7-5

(a) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c+2L$ as -5 to 5, so that $c = -5$. Then

$$\begin{aligned} a_n &= \frac{1}{L}\int_{c}^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_{0}^{5}(3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_{0}^{5}\cos\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Bigg|_{0}^{5} = 0 \qquad \text{if } n \neq 0 \end{aligned}$$

If $n = 0$, $a_n = a_0 = \frac{3}{5}\int_{0}^{5}\cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_{0}^{5} dx = 3.$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx$$

$$= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Big|_0^5 = \frac{3(1-\cos n\pi)}{n\pi}$$

(*b*) The corresponding Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}$$

$$= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right)$$

(*c*) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to $\frac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x = -5$, 0 and 5, which are points of discontinuity, the series converges to $(3+0)/2 = 3/2$ as seen from the graph. If we redefine $f(x)$ as follows,

$$f(x) = \begin{cases} 3/2 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3/2 & x = 5 \end{cases} \qquad \text{Period} = 10$$

then the series will converge to $f(x)$ for $-5 \leqq x \leqq 5$.

7.6. Expand $f(x) = x^2$, $0 < x < 2\pi$ in a Fourier series if (*a*) the period is 2π, (*b*) the period is not specified.

(*a*) The graph of $f(x)$ with period 2π is shown in Fig. 7-6 below.

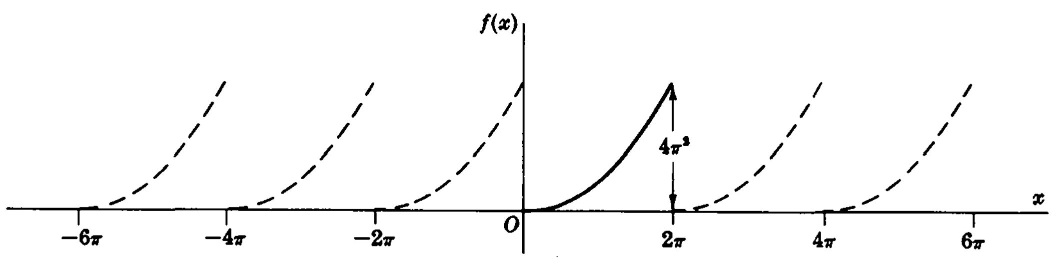

Fig. 7-6

Period $= 2L = 2\pi$ and $L = \pi$. Choosing $c = 0$, we have

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right\}\Big|_0^{2\pi} = \frac{4}{n^2}, \quad n \neq 0$$

If $n = 0$, $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{8\pi^2}{3}$.

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right\}\Big|_0^{2\pi} = \frac{-4\pi}{n}$$

Then $f(x) = x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$.

This is valid for $0 < x < 2\pi$. At $x = 0$ and $x = 2\pi$ the series converges to $2\pi^2$.

(b) If the period is not specified, the Fourier series cannot be determined uniquely in general.

7.7. Using the results of Problem 7.6, prove that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$.

At $x = 0$ the Fourier series of Problem 7.6 reduces to $\frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$.

By the Dirichlet conditions, the series converges at $x = 0$ to $\frac{1}{2}(0 + 4\pi^2) = 2\pi^2$.

Then $\frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} = 2\pi^2$, and so $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

ODD AND EVEN FUNCTIONS. HALF RANGE FOURIER SERIES

7.8. Classify each of the following functions according as they are even, odd, or neither even nor odd.

(a) $f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases}$ Period = 6

From Fig. 7-7 below it is seen that $f(-x) = -f(x)$, so that the function is odd.

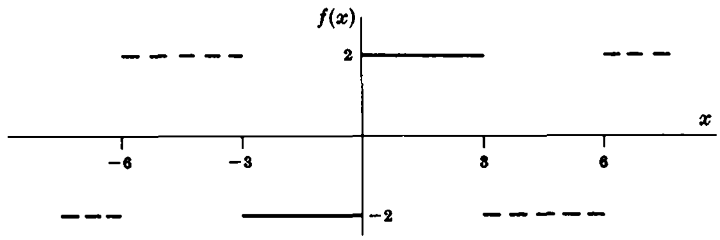

Fig. 7-7

(b) $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period = 2π

From Fig. 7-8 below it is seen that the function is neither even nor odd.

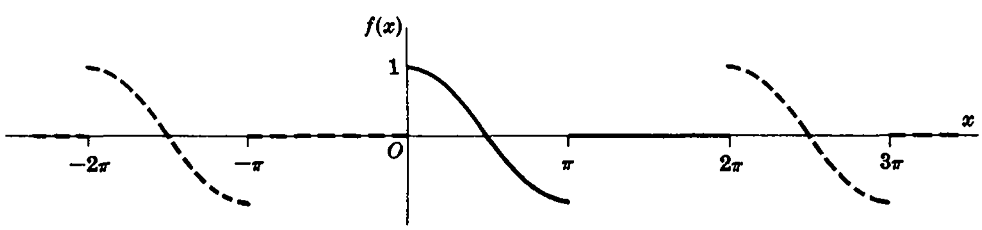

Fig. 7-8

(c) $f(x) = x(10 - x)$, $0 < x < 10$, Period = 10.

From Fig. 7-9 below the function is seen to be even.

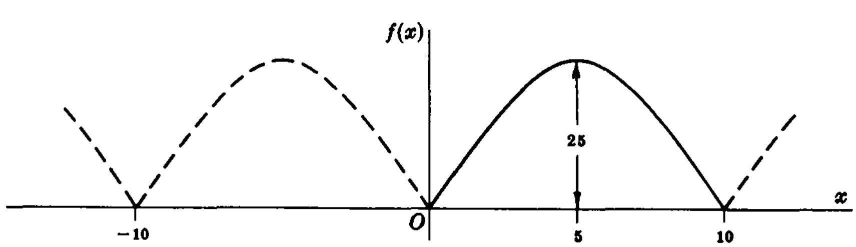

Fig. 7-9

7.9. Show that an even function can have no sine terms in its Fourier expansion.

Method 1.

No sine terms appear if $b_n = 0,\ n = 1, 2, 3, \ldots$. To show this, let us write

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \tag{1}$$

If we make the transformation $x = -u$ in the first integral on the right of (1), we obtain

$$\frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{0}^{L} f(-u)\sin\left(-\frac{n\pi u}{L}\right)du = -\frac{1}{L}\int_{0}^{L} f(-u)\sin\frac{n\pi u}{L}\,du \tag{2}$$

$$= -\frac{1}{L}\int_{0}^{L} f(u)\sin\frac{n\pi u}{L}\,du = -\frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

where we have used the fact that for an even function $f(-u) = f(u)$ and in the last step that the dummy variable of integration u can be replaced by any other symbol, in particular x. Thus from (1), using (2), we have

$$b_n = -\frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$$

Method 2.

Assume
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

Then
$$f(-x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} - b_n\sin\frac{n\pi x}{L}\right)$$

If $f(x)$ is even, $f(-x) = f(x)$. Hence

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} - b_n\sin\frac{n\pi x}{L}\right)$$

and so
$$\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L} = 0, \qquad \text{i.e.} \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}$$

and no sine terms appear.

In a similar manner we can show that an odd function has no cosine terms (or constant term) in its Fourier expansion.

7.10. If $f(x)$ is even, show that (a) $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, (b) $b_n = 0$.

(a)
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

Letting $x = -u$,

$$\frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{0}^{L} f(-u)\cos\left(\frac{-n\pi u}{L}\right)du = \frac{1}{L}\int_{0}^{L} f(u)\cos\frac{n\pi u}{L}\,du$$

since by definition of an even function $f(-u) = f(u)$. Then

$$a_n = \frac{1}{L}\int_{0}^{L} f(u)\cos\frac{n\pi u}{L}\,du + \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

(b) This follows by Method 1 of Problem 7.9.

7.11. Expand $f(x) = \sin x,\ 0 < x < \pi$, in a Fourier cosine series.

A Fourier series consisting of cosine terms alone is obtained only for an even function. Hence we extend the definition of $f(x)$ so that it becomes even (dashed part of Fig. 7-10 below). With this extension, $f(x)$ is then defined in an interval of length 2π. Taking the period as 2π, we have $2L = 2\pi$ so that $L = \pi$.

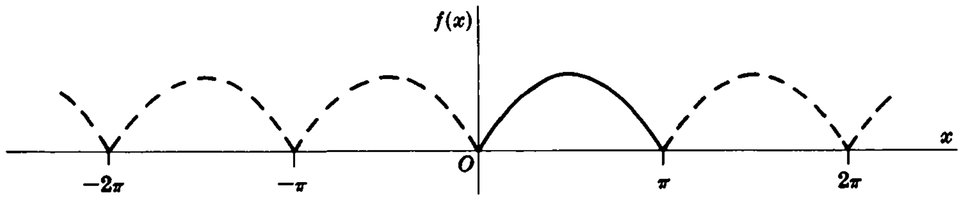

Fig. 7-10

By Problem 7.10, $b_n = 0$ and

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{\pi}\int_0^\pi \sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi \{\sin(x+nx) + \sin(x-nx)\}\,dx = \frac{1}{\pi}\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}\Bigg|_0^\pi$$

$$= \frac{1}{\pi}\left\{\frac{1-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi - 1}{n-1}\right\} = \frac{1}{\pi}\left\{-\frac{1+\cos n\pi}{n+1} - \frac{1+\cos n\pi}{n-1}\right\}$$

$$= \frac{-2(1+\cos n\pi)}{\pi(n^2-1)} \qquad \text{if } n \neq 1$$

For $n = 1$, $\quad a_1 = \frac{2}{\pi}\int_0^\pi \sin x\cos x\,dx = \frac{2}{\pi}\frac{\sin^2 x}{2}\Big|_0^\pi = 0.$

For $n = 0$, $\quad a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}(-\cos x)\Big|_0^\pi = \frac{4}{\pi}.$

Then

$$f(x) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(1+\cos n\pi)}{n^2-1}\cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right)$$

7.12. Expand $f(x) = x$, $0 < x < 2$, in a half range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4 shown in Fig. 7-11 below. This is sometimes called the *odd extension* of $f(x)$. Then $2L = 4$, $L = 2$.

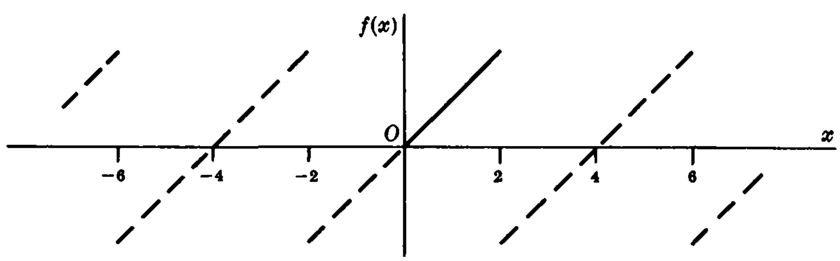

Fig. 7-11

Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then

$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

(*b*) Extend the definition of $f(x)$ to that of the even function of period 4 shown in Fig. 7-12 below. This is the *even extension* of $f(x)$. Then $2L = 4$, $L = 2$.

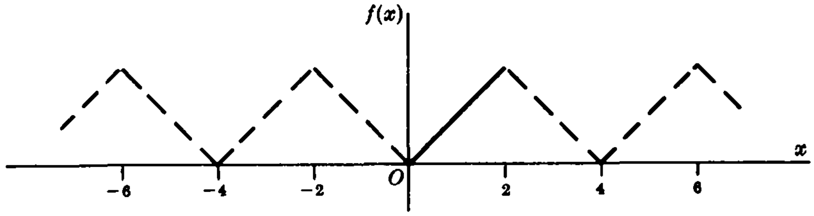

Fig. 7-12

Thus $b_n = 0$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \quad \text{if } n \neq 0$$

If $n = 0$, $a_0 = \int_0^2 x\,dx = 2$.

Then

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$$

It should be noted that the given function $f(x) = x$, $0 < x < 2$, is represented *equally well* by the two *different* series in (*a*) and (*b*).

PARSEVAL'S IDENTITY

7.13. Assuming that the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-L, L)$, prove Parseval's identity

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum(a_n^2 + b_n^2)$$

where the integral is assumed to exist.

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$, then multiplying by $f(x)$ and integrating term by term from $-L$ to L (which is justified since the series is uniformly convergent) we obtain

$$\int_{-L}^{L}\{f(x)\}^2\,dx = \frac{a_0}{2}\int_{-L}^{L}f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{1}$$

where we have used the results

$$\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx = La_n, \qquad \int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx = Lb_n, \qquad \int_{-L}^{L}f(x)\,dx = La_0 \tag{2}$$

obtained from the Fourier coefficients.

The required result follows on dividing both sides of (*1*) by L. Parseval's identity is valid under less restrictive conditions than that imposed here.

7.14. (*a*) Write Parseval's identity corresponding to the Fourier series of Problem 7.12(*b*).

(*b*) Determine from (*a*) the sum S of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\frac{1}{n^4}+\cdots$.

(*a*) Here $L=2,\ a_0=2,\ a_n=\frac{4}{n^2\pi^2}(\cos n\pi-1),\ n\neq 0,\ b_n=0.$

Then Parseval's identity becomes

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2\,dx \;=\; \frac{1}{2}\int_{-2}^{2}x^2\,dx \;=\; \frac{(2)^2}{2}+\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}(\cos n\pi-1)^2$$

or $\frac{8}{3} = 2+\frac{64}{\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)$, i.e. $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots = \frac{\pi^4}{96}$.

(*b*)
$$\begin{aligned} S &= \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots = \left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)+\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\cdots\right)\\ &= \left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)+\frac{1}{2^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right)\\ &= \frac{\pi^4}{96}+\frac{S}{16}, \quad \text{from which } S=\frac{\pi^4}{90} \end{aligned}$$

7.15. Prove that for all positive integers M,

$$\frac{a_0^2}{2}+\sum_{n=1}^{M}(a_n^2+b_n^2) \;\leqq\; \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx$$

where a_n and b_n are the Fourier coefficients corresponding to $f(x)$, and $f(x)$ is assumed piecewise continuous in $(-L,L)$.

Let
$$S_M(x) \;=\; \frac{a_0}{2}+\sum_{n=1}^{M}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right) \tag{1}$$

For $M=1,2,3,\ldots$ this is the sequence of partial sums of the Fourier series corresponding to $f(x)$.

We have
$$\int_{-L}^{L}\{f(x)-S_M(x)\}^2\,dx \;\geqq\; 0 \tag{2}$$

since the integrand is non-negative. Expanding the integrand, we obtain

$$2\int_{-L}^{L}f(x)\,S_M(x)\,dx-\int_{-L}^{L}S_M^2(x)\,dx \;\leqq\; \int_{-L}^{L}\{f(x)\}^2\,dx \tag{3}$$

Multiplying both sides of (*1*) by $2f(x)$ and integrating from $-L$ to L, using equations (*2*) of Problem 7.13, gives

$$2\int_{-L}^{L}f(x)\,S_M(x)\,dx \;=\; 2L\left\{\frac{a_0^2}{2}+\sum_{n=1}^{M}(a_n^2+b_n^2)\right\} \tag{4}$$

Also, squaring (*1*) and integrating from $-L$ to L, using Problem 7.3, we find

$$\int_{-L}^{L}S_M^2(x)\,dx \;=\; L\left\{\frac{a_0^2}{2}+\sum_{n=1}^{M}(a_n^2+b_n^2)\right\} \tag{5}$$

Substitution of (*4*) and (*5*) into (*3*) and dividing by L yields the required result.

Taking the limit as $M\to\infty$, we obtain *Bessel's inequality*

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2) \;\leqq\; \frac{1}{L}\int_{-L}^{L}\{f(x)\}^2\,dx \tag{6}$$

If the equality holds, we have Parseval's identity (Problem 7.13).

We can think of $S_M(x)$ as representing an *approximation* to $f(x)$, while the left hand side of (*2*), divided by $2L$, represents the *mean square error* of the approximation. Parseval's identity indicates that as $M\to\infty$ the mean square error approaches zero, while Bessel's inequality indicates the possibility that this mean square error does not approach zero.

The results are connected with the idea of *completeness* of an orthonormal set. If, for example, we were to leave out one or more terms in a Fourier series (say $\cos 4\pi x/L$, for example) we could never get the mean square error to approach zero no matter how many terms we took. For an analogy with 3 dimensional vectors, see Problem 7.46.

DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

7.16. (a) Find a Fourier series for $f(x) = x^2$, $0 < x < 2$, by integrating the series of Problem 7.12(a). (b) Use (a) to evaluate the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$.

(a) From Problem 7.12(a),

$$x = \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right) \qquad (1)$$

Integrating both sides from 0 to x (applying Theorem 7-2, page 184) and multiplying by 2, we find

$$x^2 = C - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \cdots\right) \qquad (2)$$

where $C = \frac{16}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$.

(b) To determine C in another way, note that (2) represents the Fourier cosine series for x^2 in $0 < x < 2$. Then since $L = 2$ in this case,

$$C = \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{2}\int_0^2 x^2\,dx = \frac{4}{3}$$

Then from the value of C in (a), we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{16}\cdot\frac{4}{3} = \frac{\pi^2}{12}$$

7.17. Show that term by term differentiation of the series in Problem 7.12(a) is not valid.

Term by term differentiation yields $2\left(\cos\frac{\pi x}{2} - \cos\frac{2\pi x}{2} + \cos\frac{3\pi x}{2} - \cdots\right)$.

Since the nth term of this series does not approach 0, the series does not converge for any value of x.

CONVERGENCE OF FOURIER SERIES

7.18. Prove that (a) $\frac{1}{2} + \cos t + \cos 2t + \cdots + \cos Mt = \frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}$

(b) $\frac{1}{\pi}\int_0^{\pi} \frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt = \frac{1}{2}, \quad \frac{1}{\pi}\int_{-\pi}^{0} \frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt = \frac{1}{2}.$

(a) We have $\cos nt \sin\frac{1}{2}t = \frac{1}{2}\{\sin(n+\frac{1}{2})t - \sin(n-\frac{1}{2})t\}$.

Then summing from $n = 1$ to M,

$$\begin{aligned}\sin\tfrac{1}{2}t\{\cos t + \cos 2t + \cdots + \cos Mt\} &= (\sin\tfrac{3}{2}t - \sin\tfrac{1}{2}t) + (\sin\tfrac{5}{2}t - \sin\tfrac{3}{2}t) \\ &\quad + \cdots + \left(\sin(M+\tfrac{1}{2})t - \sin(M-\tfrac{1}{2})t\right) \\ &= \tfrac{1}{2}\{\sin(M+\tfrac{1}{2})t - \sin\tfrac{1}{2}t\}\end{aligned}$$

On dividing by $\sin\frac{1}{2}t$ and adding $\frac{1}{2}$, the required result follows.

(b) Integrate the result in (a) from $-\pi$ to 0 and 0 to π respectively. This gives the required results, since the integrals of all the cosine terms are zero.

7.19. Prove that $\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$ if $f(x)$ is piecewise continuous.

This follows at once from Problem 7.15, since if the series $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is convergent, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

The result is sometimes called *Riemann's theorem*.

7.20. Prove that $\lim_{M\to\infty}\int_{-\pi}^{\pi} f(x)\sin(M+\frac{1}{2})x\,dx = 0$ if $f(x)$ is piecewise continuous.

We have

$$\int_{-\pi}^{\pi} f(x)\sin(M+\tfrac{1}{2})x\,dx = \int_{-\pi}^{\pi}\{f(x)\sin\tfrac{1}{2}x\}\cos Mx\,dx + \int_{-\pi}^{\pi}\{f(x)\cos\tfrac{1}{2}x\}\sin Mx\,dx$$

Then the required result follows at once by using the result of Problem 7.19, with $f(x)$ replaced by $f(x)\sin\frac{1}{2}x$ and $f(x)\cos\frac{1}{2}x$ respectively which are piecewise continuous if $f(x)$ is.

The result can also be proved when the integration limits are a and b instead of $-\pi$ and π.

7.21. Assuming that $L=\pi$, i.e. that the Fourier series corresponding to $f(x)$ has period $2L=2\pi$, show that

$$S_M(x) = \frac{a_0}{2} + \sum_{n=1}^{M}(a_n\cos nx + b_n\sin nx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

Using the formulas for the Fourier coefficients with $L=\pi$, we have

$$\begin{aligned} a_n\cos nx + b_n\sin nx &= \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos nu\,du\right)\cos nx + \left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin nu\,du\right)\sin nx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)(\cos nu\cos nx + \sin nu\sin nx)\,du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos n(u-x)\,du \end{aligned}$$

Also,
$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du$$

Then
$$\begin{aligned} S_M(x) &= \frac{a_0}{2} + \sum_{n=1}^{M}(a_n\cos nx + b_n\sin nx) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du + \frac{1}{\pi}\sum_{n=1}^{M}\int_{-\pi}^{\pi} f(u)\cos n(u-x)\,du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\left\{\frac{1}{2} + \sum_{n=1}^{M}\cos n(u-x)\right\}du \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\frac{\sin(M+\frac{1}{2})(u-x)}{2\sin\frac{1}{2}(u-x)}\,du \end{aligned}$$

using Problem 7.18. Letting $u-x=t$, we have

$$S_M(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(t+x)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

Since the integrand has period 2π, we can replace the interval $-\pi-x, \pi-x$ by any other interval of length 2π, in particular $-\pi,\pi$. Thus we obtain the required result.

7.22. Prove that

$$S_M(x) - \left(\frac{f(x+0)+f(x-0)}{2}\right) = \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{f(t+x)-f(x-0)}{2\sin\frac{1}{2}t}\right)\sin(M+\tfrac{1}{2})t\,dt + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t}\right)\sin(M+\tfrac{1}{2})t\,dt$$

From Problem 7.21,

$$S_M(x) = \frac{1}{\pi}\int_{-\pi}^{0} f(t+x)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(t+x)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt \qquad (1)$$

Multiplying the integrals of Problem 7.18(*b*) by $f(x-0)$ and $f(x+0)$ respectively,

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{\pi}\int_{-\pi}^{0} f(x-0)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(x+0)\frac{\sin(M+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt \qquad (2)$$

Subtracting (*2*) from (*1*) yields the required result.

7.23. If $f(x)$ and $f'(x)$ are piecewise continuous in $(-\pi, \pi)$, prove that

$$\lim_{M\to\infty} S_M(x) = \frac{f(x+0)+f(x-0)}{2}$$

The function $\dfrac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t}$ is piecewise continuous in $0 < t \leqq \pi$ because $f(x)$ is piecewise continuous.

Also,

$$\lim_{t\to 0+} \frac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t} = \lim_{t\to 0+} \frac{f(t+x)-f(x+0)}{t}\cdot\frac{t}{2\sin\frac{1}{2}t} = \lim_{t\to 0+}\frac{f(t+x)-f(x+0)}{t}$$

exists, since by hypothesis $f'(x)$ is piecewise continuous so that the right hand derivative of $f(x)$ at each x exists.

Thus $\dfrac{f(t+x)-f(x+0)}{2\sin\frac{1}{2}t}$ is piecewise continuous in $0 \leqq t \leqq \pi$.

Similarly, $\dfrac{f(t+x)-f(x-0)}{2\sin\frac{1}{2}t}$ is piecewise continuous in $-\pi \leqq t \leqq 0$.

Then from Problems 7.20 and 7.22, we have

$$\lim_{M\to\infty} S_M(x) - \left\{\frac{f(x+0)+f(x-0)}{2}\right\} = 0 \quad \text{or} \quad \lim_{M\to\infty} S_M(x) = \frac{f(x+0)+f(x-0)}{2}$$

ORTHOGONAL FUNCTIONS

7.24. (*a*) Show that the set of functions

$$1,\ \sin\frac{\pi x}{L},\ \cos\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{3\pi x}{L},\ \cos\frac{3\pi x}{L},\ldots$$

forms an orthogonal set in the interval $(-L, L)$.

(*b*) Determine the corresponding normalizing constants for the set in (*a*) so that the set is orthonormal in $(-L, L)$.

(*a*) This follows at once from the results of Problems 7.2 and 7.3.

(*b*) By Problem 7.3,

$$\int_{-L}^{L} \sin^2\frac{m\pi x}{L}\,dx = L, \qquad \int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,dx = L$$

Then

$$\int_{-L}^{L}\left(\sqrt{\frac{1}{L}}\sin\frac{m\pi x}{L}\right)^2 dx = 1, \qquad \int_{-L}^{L}\left(\sqrt{\frac{1}{L}}\cos\frac{m\pi x}{L}\right)^2 dx = 1$$

Also,

$$\int_{-L}^{L}(1)^2\,dx = 2L \quad \text{or} \quad \int_{-L}^{L}\left(\frac{1}{\sqrt{2L}}\right)^2 dx = 1$$

Thus the required orthonormal set is given by

$$\frac{1}{\sqrt{2L}},\ \frac{1}{\sqrt{L}}\sin\frac{\pi x}{L},\ \frac{1}{\sqrt{L}}\cos\frac{\pi x}{L},\ \frac{1}{\sqrt{L}}\sin\frac{2\pi x}{L},\ \frac{1}{\sqrt{L}}\cos\frac{2\pi x}{L},\ \ldots$$

7.25. Let $\{\phi_n(x)\}$ be a set of functions which are mutually orthonormal in (a, b). Prove that if $\sum_{n=1}^{\infty} c_n\phi_n(x)$ converges uniformly to $f(x)$ in (a, b), then

$$c_n = \int_a^b f(x)\,\phi_n(x)\,dx$$

Multiplying both sides of

$$f(x) = \sum_{n=1}^{\infty} c_n\,\phi_n(x) \tag{1}$$

by $\phi_m(x)$ and integrating from a to b, we have

$$\int_a^b f(x)\,\phi_m(x)\,dx \quad = \quad \sum_{n=1}^{\infty} c_n \int_a^b \phi_m(x)\,\phi_n(x)\,dx \tag{2}$$

where the interchange of integration and summation is justified by using the fact that the series converges uniformly to $f(x)$. Now since the functions $\{\phi_n(x)\}$ are mutually orthonormal in (a, b), we have

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx \quad = \quad \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

so that (*2*) becomes

$$\int_a^b f(x)\,\phi_m(x)\,dx \quad = \quad c_m \tag{3}$$

as required.

We call the coefficients c_m given by (*3*) the *generalized Fourier coefficients* corresponding to $f(x)$ even though nothing may be known about the convergence of the series in (*1*). As in the case of Fourier series, convergence of $\sum_{n=1}^{\infty} c_n\,\phi_n(x)$ is then investigated using the coefficients (*3*). The conditions of convergence depend of course on the types of orthonormal functions used.

Supplementary Problems

FOURIER SERIES

7.26. Graph each of the following functions and find their corresponding Fourier series using properties of even and odd functions wherever applicable.

(*a*) $f(x) = \begin{cases} 8 & 0 < x < 2 \\ -8 & 2 < x < 4 \end{cases}$ Period 4

(*b*) $f(x) = \begin{cases} -x & -4 \leqq x \leqq 0 \\ x & 0 \leqq x \leqq 4 \end{cases}$ Period 8

(*c*) $f(x) = 4x,\ 0 < x < 10,$ Period 10

(*d*) $f(x) = \begin{cases} 2x & 0 \leqq x < 3 \\ 0 & -3 < x < 0 \end{cases}$ Period 6

7.27. In each part of Problem 7.26, tell where the discontinuities of $f(x)$ are located and to what value the series converges at these discontinuities.

7.28. Expand $f(x) = \begin{cases} 2-x & 0 < x < 4 \\ x-6 & 4 < x < 8 \end{cases}$ in a Fourier series of period 8.

7.29. (*a*) Expand $f(x) = \cos x,\ 0 < x < \pi,$ in a Fourier sine series.

(*b*) How should $f(x)$ be defined at $x = 0$ and $x = \pi$ so that the series will converge to $f(x)$ for $0 \leqq x \leqq \pi$?

7.30. (*a*) Expand in a Fourier series $f(x) = \cos x,\ 0 < x < \pi$ if the period is π; and (*b*) compare with the result of Problem 7.29, explaining the similarities and differences if any.

7.31. Expand $f(x) = \begin{cases} x & 0 < x < 4 \\ 8-x & 4 < x < 8 \end{cases}$ in a series of (*a*) sines, (*b*) cosines.

7.32. Prove that for $0 \leqq x \leqq \pi$,

(*a*) $x(\pi - x) \quad = \quad \dfrac{\pi^2}{6} - \left(\dfrac{\cos 2x}{1^2} + \dfrac{\cos 4x}{2^2} + \dfrac{\cos 6x}{3^2} + \cdots\right)$

(*b*) $x(\pi - x) \quad = \quad \dfrac{8}{\pi}\left(\dfrac{\sin x}{1^3} + \dfrac{\sin 3x}{3^3} + \dfrac{\sin 5x}{5^3} + \cdots\right)$

7.33. Use Problem 7.32 to show that

(*a*) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, (*b*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$, (*c*) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} = \frac{\pi^3}{32}$.

7.34. Show that $\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}-\cdots = \frac{3\pi^3\sqrt{2}}{128}$.

DIFFERENTIATION AND INTEGRATION OF FOURIER SERIES

7.35. (*a*) Show that for $-\pi < x < \pi$,

$$x = 2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\cdots\right)$$

(*b*) By integrating the result of (*a*), show that for $-\pi \leqq x \leqq \pi$,

$$x^2 = \frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\cdots\right)$$

(*c*) By integrating the result of (*b*), show that for $-\pi \leqq x \leqq \pi$,

$$x(\pi-x)(\pi+x) = 12\left(\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\cdots\right)$$

7.36. (*a*) Show that for $-\pi < x < \pi$,

$$x\cos x = -\frac{1}{2}\sin x + 2\left(\frac{2}{1\cdot 3}\sin 2x - \frac{3}{2\cdot 4}\sin 3x + \frac{4}{3\cdot 5}\sin 4x - \cdots\right)$$

(*b*) Use (*a*) to show that for $-\pi \leqq x \leqq \pi$,

$$x\sin x = 1-\frac{1}{2}\cos x - 2\left(\frac{\cos 2x}{1\cdot 3}-\frac{\cos 3x}{2\cdot 4}+\frac{\cos 4x}{3\cdot 5}-\cdots\right)$$

7.37. By differentiating the result of Problem 7.32(*b*), prove that for $0 \leqq x \leqq \pi$,

$$x = \frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right)$$

PARSEVAL'S IDENTITY

7.38. By using Problem 7.32 and Parseval's identity, show that

$$(a)\ \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \qquad (b)\ \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$$

7.39. Show that $\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\cdots = \frac{\pi^2-8}{16}$. [*Hint.* Use Problem 7.11.]

7.40. Show that (*a*) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$, (*b*) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$.

7.41. Show that $\frac{1}{1^2\cdot 2^2\cdot 3^2}+\frac{1}{2^2\cdot 3^2\cdot 4^2}+\frac{1}{3^2\cdot 4^2\cdot 5^2}+\cdots = \frac{4\pi^2-39}{16}$.

ORTHOGONAL FUNCTIONS

7.42. Given the functions a_0, a_1+a_2x, $a_3+a_4x+a_5x^2$ where $a_0,\ldots,a_5$ are constants. Determine the constants so that these functions are mutually orthonormal in $(-1,1)$ and thus obtain the functions.

7.43. Generalize Problem 7.42.

7.44. (*a*) Show that the functions $\frac{e^{im\phi}}{\sqrt{2\pi}}$, $m = 0, \pm 1, \pm 2, \ldots$ are mutually orthonormal in $(-\pi,\pi)$. (*b*) Show how to expand a function $f(x)$ in a series of these functions and explain the connection with Fourier series.

7.45. Let $f(x)$ be approximated by the sum of the first M terms of an orthonormal series

$$\sum_{n=1}^{M} c_n\phi_n(x) = S_M(x)$$

where the functions $\phi_n(x)$ are orthonormal in (a,b). (*a*) Show that

$$\int_a^b [f(x)-S_M(x)]^2\,dx = \int_a^b [f(x)]^2\,dx - \sum_{n=1}^{M} c_n^2$$

(*b*) By interpreting

$$\frac{1}{b-a}\int_a^b [f(x) - S_M(x)]^2\,dx$$

as the *mean square error* of $S_M(x)$ from $f(x)$ [and the square root as the *root mean square* or *r.m.s.* error], show that Parseval's identity is equivalent to the statement that the root mean square error approaches zero as $M \to \infty$.

(*c*) Show that if the root mean square error may not approach zero as $M \to \infty$, then we still have Bessel's inequality

$$\sum_{n=1}^{\infty} c_n^2 \leqq \int_a^b [f(x)]^2\,dx$$

(*d*) Discuss the relevance of these results to Fourier series.

7.46. Let **r** be any three dimensional vector. Show that

$$(a)\quad (\mathbf{r}\cdot\mathbf{i})^2 + (\mathbf{r}\cdot\mathbf{j})^2 \leqq \mathbf{r}^2 \qquad (b)\quad (\mathbf{r}\cdot\mathbf{i})^2 + (\mathbf{r}\cdot\mathbf{j})^2 + (\mathbf{r}\cdot\mathbf{k})^2 = \mathbf{r}^2$$

and discuss these with reference to Bessel's inequality and Parseval's identity. Compare with Problem 7.15.

7.47. Suppose that one term in any orthonormal series [such as a Fourier series] is omitted. (*a*) Can we expand a function $f(x)$ into the series? (*b*) Can Parseval's identity be satisfied? (*c*) Can Bessel's inequality be satisfied? Justify your answers.

7.48. Let $\{\phi_n(x)\}$, $n = 1, 2, 3, \ldots$, be orthonormal in (a, b). Prove that

$$\int_a^b \left[f(x) - \sum_{n=1}^{M} c_n\,\phi_n(x) \right]^2 dx$$

is a minimum when

$$c_n = \int_a^b f(x)\,\phi_n(x)\,dx$$

Discuss the connection of this to (*a*) Fourier series and (*b*) Problem 7.45.

7.49. (*a*) Show that the functions $1, 1 - x, 2 - 4x + x^2$ are mutually orthogonal in $(0, \infty)$ with respect to the density function e^{-x}. (*b*) Obtain a mutually orthonormal set.

7.50. Give a vector interpretation to functions which are orthonormal with respect to a density or weight function.

Answers to Supplementary Problems

7.26. (*a*) $\dfrac{16}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n}\sin\frac{n\pi x}{2}$ (*b*) $2 - \dfrac{8}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{(1-\cos n\pi)}{n^2}\cos\frac{n\pi x}{4}$

(*c*) $20 - \dfrac{40}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{5}$ (*d*) $\dfrac{3}{2} + \displaystyle\sum_{n=1}^{\infty}\left\{\frac{6(\cos n\pi - 1)}{n^2\pi^2}\cos\frac{n\pi x}{3} - \frac{6\cos n\pi}{n\pi}\sin\frac{n\pi x}{3}\right\}$

7.27. (*a*) $x = 0, \pm 2, \pm 4, \ldots; 0$ (*b*) no discontinuities

(*c*) $x = 0, \pm 10, \pm 20, \ldots; 20$ (*d*) $x = \pm 3, \pm 9, \pm 15, \ldots; 3$

7.28. $\dfrac{16}{\pi^2}\left\{\cos\dfrac{\pi x}{4} + \dfrac{1}{3^2}\cos\dfrac{3\pi x}{4} + \dfrac{1}{5^2}\cos\dfrac{5\pi x}{4} + \cdots\right\}$

7.29. (*a*) $\dfrac{8}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{n\sin 2nx}{4n^2-1}$ (*b*) $f(0) = f(\pi) = 0$

7.30. Same answer as in Problem 7.29.

7.31. (*a*) $\dfrac{32}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{8}$ (*b*) $2 + \dfrac{16}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\left(\frac{2\cos n\pi/2 - \cos n\pi - 1}{n^2}\right)\cos\frac{n\pi x}{8}$